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Differential Geometric notions in Particle Physics

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Abstract

The purpose of this work is to study the relationship between various areas of contemporary differential geometry and particle physics. Modern theoretical physics is nowadays an experimental arena where a wide variety of geometrical structures are studied to try and capture the properties of nature at a fundamental level.

We will go from the more concrete notions of 19th century differential geometry to some more abstract structures that were developed during the 20th century to capture the fundamental properties of spaces. Once we have this tools, we will devote to the study of complex differential geometry, emphasizing the important role played by *Hermitian structures*, which are also fundamental for understanding Quantum behaviour in geometric settings.

Kähler manifolds, specially compact ones, will also play a very important role in the last chapter, where we will see three different branches of geometry meet, Riemannian geometry, Symplectic geometry and Complex geometry. To understand those, a study of representations of the *Lie Algebra* $\mathfrak{sl}(2,\mathbb{C})$ will be undertaken.

Afterwards we will apply the geometric notions developed to certain problems in embedding theory and sketch Yau's proof of the Calabi Conjecture. This conjecture was of fundamental importance for *String Theory*, where *Calabi-Yau manifolds* are a fundamental geometric notion for the theory.

1 Manifolds and bundles

Many classes of manifolds are of interest to both the geometer and the physicist, and they are studied from a variety of distinct points of view. We will be mainly interested, during this work, in two special classes, *differentiable manifolds* and *complex manifolds*. The main things to study when dealing with this class of objects are

- The geometric/topological properties of the manifolds.
- The analysis of functions/sections that are defined on manifolds (this is at the core of modern theoretical physics).
- The two-side interaction between the previous points, and this interaction is of fundamental importance for both geometry and physics.

In this section we will briefly review some of the differential-geometric notions that will be needed in later chapters so as to start from a common ground (and also to introduce the notation that will be used).

The perspective we will mainly take, is to apply the knowledge and techniques from real analysis to problems arising in the theory of complex manifolds.

We will divide this chapter in three main topics, first we will review some fundamental definitions about differential geometry, to introduce the notation that will be used during later chapters and to get a "common ground" for the ideas that will later appear.

Later, we will review a bit of the theory of bundles, which are of fundamental importance to both mathematicians and theoretical physicists. In fact it was a big revolution in particle physics during the 70's to understand that what physicists were studying in Gauge theory was the same that differential geometers had been studying for more than 50 years, under the name of connections in fibre bundles. There is a famous paper [61] that was the first step into building this dictionary between mathematical notions and physics. From there on the mutual relationship between both subjects has been very productive.

Reviewing the theory of bundles will also be of fundamental importance when we in the next chapter develop the theory of sheaves, since in a sense one can view them as a big generalization of bundles, and hence as a fundamental notion for modern geometry, both differential, complex and algebraic. Sheaf theory is not used a lot in usual real differential geometry due to the existence of partitions of unity, but when one works in the complex or algebraic settings, they become an indispensable tool, as we will later see.

Afterwards we will give some notions about what are usually called almost-complex structures/manifolds, and their associated $\overline{\partial}$ operators, that will be of fundamental importance during later chapters, where we will deal with *complex differential geometry*.

1.1 Manifolds

In this section we will review quickly the usual definitions that come to play in the category of manifolds that we are interested in, manifolds, morphisms... We will try to give a common view of many geometrical situations, due to the fact that the definitions are mainly always the same, but what varies is the class of functions that one wants to do analysis on. But the important thing to remark is that the ideas behind them are very similar, and all of them are no more than the formal realizations of Riemann's revolutionarization of geometry in his doctoral thesis [53]. We will try to give some examples as we go of manifolds that are of interest both to geometers and physicists.

We will only work with two base fields, either the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$, both with their usual metrics and topologies. We will denote by $\mathbb{K}$ the ground field that we are using, one of the previous two. Now, if $U \subset \mathbb{K}^n$ is an open subset we will mainly be concerned with the following classes of functions:

1. In the real setting, meaning $\mathbb{K} = \mathbb{R}$:
   (a) $\mathcal{E}(U)$ will denote the infinitely differentiable functions on U. These functions are usually called *smooth*, and usually denoted by $\mathcal{C}^\infty(U)$. We will use during the work both of this notations, depending on the setting. If one wants to restrict the degree of regularity of the functions, one also can consider spaces like $\mathcal{C}^k(U)$ and many variations of these, that we will use in later chapters during the work, mainly when we have to deal with the *Calabi conjecture*.
   (b) $\mathcal{A}(U)$ will denote the real-valued analytic functions on U. Meaning that $\mathcal{A}(U) \subset \mathcal{E}(U)$, and $f \in \mathcal{A}(U)$ if the Taylor expansion converges around every point in U.
2. In the complex setting, meaning $\mathbb{K} = \mathbb{C}$:
   (a) $\mathcal{O}(U)$ will denote the complex-valued holomorphic functions on U. And in the complex setting, as one knows from complex analysis, there are many properties of holomorphic functions that distinguish them from smooth functions or real-analytic ones, this distinctions will come into play in geometry.

We start by defining the common ground for all of the geometric structures that we will be interested in studying in these chapter.

Definition 1.1.1. A *topological $k-$ manifold* is a Hausdorff topological space with a countable basis which is locally homeomorphic to some open set of $\mathbb{R}^n$. The integer n is called the *topological dimension* of the manifold.

Remark. We will work under the assumption that our manifolds are connected.

Now we will make a common definition for all the geometric manifolds that we will study. To do this we will denote by $\mathcal{S}$ the class of $\mathbb{K}-$valued functions defined on open sets of $\mathbb{K}^n$ that we described above. This just means that when we say $\mathcal{S}(U)$ for some open set $U \subset \mathbb{K}^n$ we will mean some of the classes of functions introduced before. With this remark in our head, we make the following definition.

Definition 1.1.2. We say that $\mathcal{S}_M$ is an $S-$*structure* on a $\mathbb{K}-$manifold M if it is a family of $\mathbb{K}-$values functions defined on the open sets of our topological manifold M satisfying

1. For every point $p \in M$ there is an open neighborhood $p \in U$ and a homeomorphism $\varphi\colon U \to U'$, where $U' \subset \mathbb{K}^n$ is an open set. We require that for any other open set $V \subset U$
$$f\colon V \to \mathbb{K} \in \mathcal{S}_M \iff f \circ \varphi^{-1} \in \mathcal{S}(\varphi(V)).$$
2. If we have $f\colon U \to \mathbb{K}$, where $U = \cup_i U_i$ is a union of open sets U_i, then
$$f \in \mathcal{S}_M \iff f|_{U_i} \in \mathcal{S}_M \forall i.$$

From (1.) it clearly follows that if we are working over the real numbers, meaning $\mathbb{K} = \mathbb{R}$, then the dimension of the underlying topological manifold k is just equal to n, and if we are working over the complex numbers, then $k = 2n$. Meaning that over the complex numbers we are always working on even real dimension. In either case, the positive integer n will be called the $\mathbb{K}-$*dimension* of M. We call a manifold with an $\mathcal{S}-$structure a $\mathcal{S}-$manifold, and is usually denoted by $(M, \mathcal{S}_M)$. The homeomorphisms $\varphi\colon U \to U' \subset \mathbb{K}^n$ are called *coordinate systems*, and usually if one thinks of them as being coordinates one uses the inverse map φ^{-1}.

Remark. Now, if we apply this abstract definition to the concrete case of a certain class of functions, we see that we have defined the usual objects of (differential) geometry:

1. If $\mathcal{S} = \mathcal{E}$: then the definition is just the usual definition of a $\mathcal{C}^\infty$ manifold, and the functions in $\mathcal{E}_M$ are usually called smooth functions on the manifold.
2. If $\mathcal{S} = \mathcal{A}$: then the definition is just the same but for real-analytic manifolds.
3. If $\mathcal{S} = \mathcal{O}$: then the definition is just the usual definition of a complex manifold, and the functions $\mathcal{O}_M$ are called holomorphic functions on M. Note that by taking the same definition but with meromorphic functions (those having *Laurent expansions*) we can also study meromorphic functions on manifolds, meaning holomorphic functions almost everywhere but with allowing for some singularities.

Now that we have defined the objects in our appropriate categories (manifolds), we want to define (as usual in mathematics) the appropriate morphisms between them. In this general setting we also define morphisms in a more abstract way to incorporate all possible class of functions that one may be interested in while dealing with geometric problems.

Definition 1.1.3. If $(M, \mathcal{S}_M)$ and $(N, \mathcal{S}_N$ are manifolds, then we say that a continuous map $F\colon M \to N$ is a $\mathcal{S}-$*morphism* if
$$f \in \mathcal{S}_N \iff f \circ F \in \mathcal{S}_M.$$

Furthermore, if the map $F\colon M \to N$ is a homeomorphism and F^{-1} is also a $\mathcal{S}-$morphism, then we say that F is a $\mathcal{S}-$*isomorphism*.

Remark. One should note here that it follows from this definitions that if on a certain $\mathcal{S}-$manifold $(M, \mathcal{S}_M)$ we have two coordinates

$$\varphi_1\colon U_1 \to \mathbb{K}^n$$
$$\varphi_2\colon U_2 \to \mathbb{K}^n$$

that have a non-empty overlap, meaning that $U_1 \cap U_2 \neq \phi$, then

$$\varphi_2 \circ \varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2),$$

is an $\mathcal{S}-$isomorphisms of open subsets of $(\mathbb{K}^n, \mathcal{S}_{\mathbb{K}^n})$, which is the more common definition one has for morphisms in classical differential geometry.

Conversely, if we have an open covering $\{U_\alpha\}_\alpha$ of a topological manifold M, and a family of homeomorphisms $\{\varphi_\alpha\colon U_\alpha \to U_{\alpha'} \subset \mathbb{K}^n\}_\alpha$, satisfying the compatibility condition above, then this clearly defines an $\mathcal{S}-$structure the way we defined. Hence this definition is just a more abstract and general way of looking at the usual definitions. In this setting the collection $\{(U_\alpha, \varphi_\alpha)\}$ is usually called an *atlas* for the manifold M. If we unravel this definition for some concrete class of functions we get the usual notions, namely:

1. If $\mathcal{S} = \mathcal{E}$: then this is just the notion of a *differentiable mapping* and *diffeomorphism* between two manifolds.

2. If $\mathcal{S} = \mathcal{A}$: then this is the notion of a *real-analytic map* and of *real-analytic isomorphism*, sometimes called *bianalytic*.

3. If $\mathcal{S} = \mathcal{E}$: then this is the notion of *holomorphic map* between manifolds, and the appropriate notion of isomorphism.

Thus with this definitions one can do differential geometry over the class of functions that is more appropriate to the problem one has to deal with. And as we said, this abstract definitions are a shorter way of defining the classical objects that one deals with in differential geometry, analytic geometry, complex geometry...

It is clear that one can restrict the structure (class of functions) that one has to some subset, and this ends up in the usual notion of a *submanifold*. One should note that due to the fact that the implicit function theorem works in all of the classes of functions that we have talked about, one can also understand submanifolds as being locally the zero-set of some family of functions. And this is the more classical (XIX century) view on manifolds, that was mainly used and is very useful for getting intuition about what a manifold can be. And although many manifolds arise this way, introducing the abstract notion of a manifold was fundamental for getting an *intrinsic* view of geometry. This was mainly introduced into mathematics (the idea, not the formal definition) by the german mathematician Bernhard Riemann, building on the previous work done by many french mathematicians and by Gauss. One should note that one of the main motivations to introduce this abstract notion of manifolds was a theorem by Gauss, that he called *Theorema Egregium* ("Remarkable theorem"). This theorem tells us that *curvature*, which was thought of as being a property of objects inside some ambient space, was in fact something *intrinsic* (it is an invariant under a local isometry). The term "manifold" comes in fact from Riemann's thesis, where he introduced the german word *Mannigfaltigkeit*, to describe the intuitive ideas of how geometry should be thought of. His thesis contributed greatly to the understanding of both geometry and the physical world, and even Riemann was close to propose ideas similar to General Relativity, since with metrics one can accurately describe curved spaces. His problem here, and the big revolution introduced by Einstein, was that one should think of time just as another dimension in the manifold describing the world.

Riemann had the intuitive idea that $n-$manifolds (equipped with a metric) are just the appropriate objects for describing arbitrary quantities that depend on $n-$parameters, and he used this ideas to change forever the field of complex analysis via the introduction of *Riemann surfaces*. His ideas gave a much more geometric view into problems that one encounters in "Cauchy-like" complex analysis, like multivalued-functions, branches, analytic continuation...

Once the work of Riemann started to be understood in the mathematical community of the time, several mathematicians started to work on this ideas. But one should note that they **didn't have the formal definition** that forms the first page of almost every differential-geometry book nowadays. It is important to note the big amounts of intuition they had, and for instance the french mathematician Henri Poincaré would be one of the prototypical examples. He praised intuition almost above anything else, without disregarding logic and proofs, something that from the point of view of the author should something to consider in the nowadays "Bourbaki-like" formal mathematics, where intuition is just something optional that one may or may not build. Quoting Poincaré in his book *Science et méthode*:

C'est par la logique qu'on démontre, c'est par l'intuition qu'on invente.
(It is by logic that we prove, but by intuition that we discover.)

La logique nous apprend que sur tel ou tel chemin nous sommes sûrs de ne pas rencontrer d'obstacle ; elle ne nous dit pas quel est celui qui mène au but. Pour cela il faut voir le but de loin, et la faculté qui nous apprend à voir, c'est l'intuition. Sans elle, le géomètre serait comme un écrivain qui serait ferré sur la grammaire, mais qui n'aurait pas d'idées.

(Logic teaches us that on such and such a road we are sure of not meeting an obstacle; it does not tell us which is the road that leads to the desired end. For this, it is necessary to see the end from afar, and the faculty which teaches us to see is intuition. Without it, the geometrician would be like a writer well up in grammar but destitute of ideas.)

Afterwards, the german mathematician Hermann Weyl in 1912 gave one of the first formal *intrinsic* definition of a manifold. And later the formal theory of manifolds and Lie Groups was built by the american mathematician Hassler Whitney and others, their work clarified many of the ideas and results that were produced by the mathematicians that preceded them. And with his *embedding theorem*, that we will state, Whitney was able to relate both intrinsic and extrinsic perspectives.

There are a huge amount of examples of manifolds appearing both in mathematics and in physics, and this is the reason that they are a necessary object in both fields. Spaces such as *Euclidean space,Projective space, Grassmanians, Algebraic submanifolds...* are some usual examples of manifolds appearing both in mathematics and physics.

We will now give a definition of what it means for a manifold to be *embedded* in another manifold. Although it seems like an easy definition (and it is), characterizing the manifolds that admit such embeddings into, for instance, Euclidean space or Projective space, is not at all an easy task. And the important thing to note here is that the answer to this question highly depends on the class of manifolds, things are not the same (and in fact very different) in the real case and in the complex case. Embeddings of real manifolds into euclidean space was a problem solved by Whitney, as we said. But the same question for complex manifolds was only solved 30 or 40 years later, and it required the use of a lot of theoretical machinery to be able to solve the problem. In the case of embeddings of complex manifolds into Euclidean (complex) space, the problem was solved by the french mathematician Henri Cartan (the son of Élie Cartan) as we will see in later chapters. The related problem of embedding into (complex) Projective space was solved later by the japanese mathematician Kunihiko Kodaira, as we will also see in later chapters.

Definition 1.1.4. We say that an $\mathcal{S}-$morphism

$$f\colon (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$$

between to $\mathcal{S}-$manifolds is an $\mathcal{S}-$*embedding* if f is an $\mathcal{S}-$isomorphism onto a certain submanifold of the image.

As we said, embeddings will be the first place where we start to see some of the big differences between certain class of manifolds, mainly between real (smooth) and complex manifolds.

As we said, there is a famous theorem due to Whitney, that relates, **in the real case** both notions of a manifold, intrinsic and extrinsic. This theorem, which is usually called *Whitney's embedding theorem* goes as follows.

Theorem 1.1.1. *If M is a differentiable n-manifold. Then there exists an embedding (differentiable) of the manifold M into the Euclidean space $\mathbb{R}^{2n+1}$. And moreover, the image of M under this embedding can be realized as a real-analytic submanifold of $\mathbb{R}^{2n+1}$.*

As we said, this result give us precisely that any differentiable manifold (without any topological restriction) can be embedded into Euclidean space. Hence, the more classical perspective for real manifold is in fact, due to the theorem, the same as the more abstract intrinsic one. And even though the intrinsic perspective is very useful in many areas, both in mathematics and physics, one could think (and with good reason) that then there should be "no need" for the abstract definition, since the classical old way of looking at manifolds just as subsets of Euclidean space is enough to deal with every differentiable manifold. There is also a deeper and harder result [62] that tells us that the same story is in a sense true for real-analytic manifolds. And thus one is further convinced of the idea that those previous definitions were just a "mathematical entertainment" that may be of use to get the intrinsic perspective but no more. And that is kind of true in the real case, but if one looks at complex manifolds, the story radically changes. But why? To see this, we will start with an elementary result that will tell us the big difference between real and complex manifolds. As we saw, both compact and non-compact real manifolds admit embeddings into Euclidean space, but does the same hold for complex manifolds? And the answer to this question is a radical no, as we see in the following theorem.

Theorem 1.1.2. *If X is a connected* ***compact*** *complex manifold and we take an arbitrary holomorphic function $f \in \mathcal{O}(X)$. Then f is just a constant, meaning that the only global holomorphic functions are constants.*

Proof. Take an arbitrary holomorphic function $f \in \mathcal{O}(X)$. The main topological restriction that we have here is compactness, thus we will need to use that. Since f is continuous on X and it is compact, then we know that $|f|$ assumes its maximum on X, for example at a point $x_0 \in X$. Due to continuity, we also have that the set $S = \{x\colon f(x) = f(x_0)\}$ is closed inside X. Of course, we know that the set S is also non-empty since $x_0 \in S$.

Now one should think, is S also open? Because if that is the case, then by the usual argument $S = X$ and the function f is just a constant. We will now see that this is in fact the case. To do this, take an arbitrary point $x \in S$ and consider local coordinates $z = (z_1, ..., z_n)$ around the point x with $z = 0$ corresponding to the point x. Now, if we consider a small ball B around $z = 0$ and take some $z \in B$, we can then define the function $g(\lambda) = f(\lambda z)$, an ordinary function of one complex variable λ assuming it's maximum value at $\lambda = 0$. One can thus apply the maximum principle for ordinary holomorphic functions of a single variable to conclude that g must be constant. Thus concluding that $f(z) = f(0) \forall z \in B$, and this means that in fact $B \subset S$. Hence we have checked that S is in fact open, and by what we said before this implies that f must be a constant function. □

Remark. One should note here that the proof can be made a little quicker if one knows that the maximum principle for holomorphic functions is also true in domains in $\mathbb{C}^n$, thus applying it directly without doing the trick we did for obtaining a single-variable holomorphic function.

From this theorem we deduce that the situation for compact complex manifolds has to be completely different than the real situation, where there are lots of globally defined smooth functions usually. What does this theorem tells us about embedding compact complex manifolds into $\mathbb{C}^n$?

Corollary. There are no compact complex submanifolds of $\mathbb{C}^n$ of non-zero dimension.

Proof. This is clear since otherwise the coordinate functions $z_1, ..., z_n$ would be globally defined non-constant holomorphic functions, thus contradicting the previous theorem. □

We deduce from this that the problem of embeddings changes radically in the complex case. To start, there are topological restrictions (compactness) for a manifold to be embedded into Euclidean (complex) space. Thus if one is interested in the question of which complex manifolds admit an embedding into $\mathbb{C}^n$, then one should focus the attention on non-compact manifolds. But if one is interested in finding embeddings of compact complex manifolds, what this results tell us is that we should embed our manifold into another space, and the usual candidate here is $\mathbb{CP}^n$. One can in fact characterize both classes of complex manifolds, those admitting an embedding into $\mathbb{C}^n$, which are called *Stein manifolds.* And also one can characterize those compact complex manifolds admitting an embedding into $\mathbb{CP}^n$. In the following chapter we will see how one deals with this questions, the main theoretical tool to deal with those questions will be *Sheaf theory.* This theory, of which we will talk about later, is not very important in the case of real manifolds (due to the existence of partitions of unity) but in the case of complex manifolds it becomes an indispensable tool. Those sheaf-theoretic notions later where used mainly by french mathematicians (and specially Grothendieck) to revolutionize the whole field of Algebraic Geometry. Since we will have an analytic approach to geometry, we will not devote to the study of algebraic techniques, but the interested reader is referred to [64] for more information.

Due to the fact that we will be interested in the problem of embeddings into projective space, we make the following definition.

Definition 1.1.5. A compact complex manifold X which admits an embedding into $\mathbb{CP}^n$ (for some n) is called a *projective algebraic manifold.*

Remark. The reader should note that we made a small "trick" here, in the name of the object. One clearly understands why we call it manifold and projective, that is obvious from the definition. But why have we called them algebraic? This is highly non-obvious from the definitions that we made, what does the word algebraic mean here? It actually means that the embedded manifold can be expressed as the zeros of homogeneous polynomials in homogeneous coordinates. But following our definitions, that is not a trivial thing to conclude, and that is true. It is in fact a theorem by the chinese mathematician Wei-Liang Chow [48] that this is actually the case. This means, as we said, that techniques from algebraic geometry can be applied to the study of these manifolds. Since we will not be using these algebro-geometric techniques in our work, we just give this as a mention for the interested reader, but we will not prove Chow's theorem. The relationship between both perspectives (analytic and algebraic) goes a lot further than that, and was deeply explored by the french mathematician Jean-Pierre Serre during the 50's. The interested reader can check [50] for more information about this important relationship.

1.2 Vector Bundles

In this section we will briefly introduce the notion of *bundles*, and specially *vector bundles*, which are a fundamental notion both in differential geometry and in physics. Their use has had a profound impact in many areas of math, and it can not be overstated. Almost every aspect of differential geometry deals, in one way or another, with the notion bundles. Their role is to generalize the usual cartesian product construction, to be able to twist this product topologically in non-trivial ways. The usual example here is the difference between a cylinder and a Möbius band.

The study of vector bundles is primairly motivated by the desire to linearize nonlinear problems in geometry, and every mathematician knows the importance of linearizing problems. Linear Algebra is by far one of the branches of math that we know better, and hence reducing problems to it is a very useful thing to do, since it enables us to use the machinery of linear algebra to deal with problems.

From the physics point of view, as was said before, bundles are a fundamental part of modern particle physics. In the 70's, physicists realized that the objects that they use to describe the fundamental interactions of nature, what they call *Gauge fields*, were in fact no more than connections in certain bundles over space-time. And those connections had been under the intensive study of differential geometers for more than 40 years, thus connecting both was revolutionary.

We will use the same notation as before, and thus $\mathcal{S}$ will denote one of the three structures on manifolds (smooth, analytic or holomorphic). And the ground field $\mathbb{K}$ will also be either $\mathbb{R}$ or $\mathbb{C}$. We will now give the definition of a *vector bundle.*

Definition 1.2.1. Let $\pi\colon E \to X$ be a continuous map between two Hausdorff spaces. This structure is called a $\mathbb{K}$-*vector bundle of rank* r if it satisfies:

1. $E_p\colon = \pi^{-1}(p)$ for $p \in X$ is a $\mathbb{K}-$vector space of dimension r. We call E_p the *fibre* or *fiber* over the point p.

2. For every point $p \in X$, there is a neighborhood $p \in U$ and a homeomorphism

$$\varphi\colon \pi^{-1}(U) \to U \times \mathbb{K}^r,$$

satisfying $\varphi(E_p) \subset \{p\} \times \mathbb{K}^r$, and the map φ^p defined by

$$\varphi^p\colon E_p \xrightarrow{\varphi} \{p\} \times \mathbb{K}^r \xrightarrow{proj.} \mathbb{K}^r,$$

is a $\mathbb{K}-$ vector space isomorphism. We call this structure a *local trivialization.*

Remark. For a certain $\mathbb{K}-$vector bundle $\pi\colon E \to X$, we usually say that E it the *total space* and X the *base space*. The usual terminology is that E is a vector bundle over X.

Note that for any two local trivializations (U_1, φ_1) and (U_2, φ_2) we can consider

$$\varphi_1 \circ \varphi_2^{-1}\colon (U_1 \cap U_2) \times \mathbb{K}^r \to (U_1 \cap U_2) \times \mathbb{K}^r,$$

which induces a map

$$g_{12}\colon U_1 \cap U_2 \to GL(r, \mathbb{K}),$$

where

$$g_{12}(p) = \varphi_1^p \circ (\varphi_2^p)^{-1}\colon \mathbb{K}^r \to \mathbb{K}^r.$$

The functions g_{12} for every possible pair of local trivializations are called the *transition functions* of the bundle. Note that the order is significant.

One can check that this transition functions satisfy the compatibility conditions

$$g_{12}g_{23}g_{31} = Id_r \quad \textit{on } U_1 \cap U_2 \cap U_3,$$

and

$$g_{11} = Id_r \quad \textit{on } U_1,$$

where the product we are considering is just usual matrix multiplication and Id_r the identity matrix (of rank r).

Note that we have defined bundles just in terms of the underlying topologies and using continuity. If one is working under a class of functions having more properties than just continuity, as we did in the previous section, one has to adapt the definition to the class of functions. This is done in the following definition.

Definition 1.2.2. We say that a $\mathbb{K}-$vector bundle of rank r

$$\pi\colon E \to X$$

is an $\mathcal{S}-$bundle if both E and X are $\mathcal{S}-$manifolds, π is an $\mathcal{S}-$morphism, and the local trivialization functions are $\mathcal{S}-$isomorphisms.

Remark. One should note here that if one is given an $\mathcal{S}-$manifold, and open covering such that for each ordered non-empty intersection one has an $\mathcal{S}-$function satisfying the compatibility conditions mentioned above, then one can construct a vector bundle having these transition functions. This construction can be seen for instance in [1].

We said before that bundles are a kind of generalization of cartesian products, but why? Under our definitions, this is not as clear, due to the fact that we have required our fibers to be vector spaces, and thus we are restricted to just some special cartesian products. For instance for any $\mathcal{S}-$manifold M, one can consider

$$\pi\colon M \times \mathbb{K}^r \to M,$$

with π being the natural projection. This is clearly a bundle an is called the *trivial bundle*.

One could in fact define a more general class of objects by requiring that the fiber to be another object of interest. One can consider for instance bundles were the fiber is just another manifold, and here is where one sees that the notion of a bundle fully generalizes the notion of cartesian product of two manifolds. Of special interest in mathematics and physics are those bundles having as fibers what are called *Lie Groups* (groups with some compatible manifold structure), and those are called *principal bundles*. Lie Groups form the basis of our current understanding of nature (in the form of the standard model), which can be seen as trying to construct theories that are invariant under certain compact Lie groups. They can also be applied to the study of the dynamics of solids, and are fundamental for understanding symmetries both in geometry and physics. Special examples of those groups will appear in later chapters, and we will be specially interested in their *representations* (how the groups can act as endomorphisms of certain vector spaces). This representations are also of fundamental importance in physics, since due to the work of many physicists and mathematicians during last century we know that the way to classify particles is via irreducible representations. This idea was first introduced by the Hungarian physicist Eugene Wigner, and has stayed in physics since it was introduced.

Example 1.1. One of the other natural examples of a vector bundle is the so-called *tangent bundle*. If M is a differentiable manifold, this bundle formalizes the notion of linearization of the manifold. At every fiber we want to have a linearization of the underlying manifold.

To do this, for every $p \in M$, one considers

$$\mathcal{E}_{M,p} = \varinjlim_{p \in U \subset M} \mathcal{E}_M(U),$$

for $U \subset M$ running over open subsets containing the point p. This algebra (over $\mathbb{R}$) is called the *algebra of germs of differentiable functions at the point $p \in M$*. The inductive (or direct) limit is taken with respect to the natural inclusions. This construction can be done in a much more general setting, in sheaf theory, as we will later see.

What this abstract construction tells us, in more concrete terms, is that two smooth functions f and g defined near the point p agree on some open neighborhood of p, then they represent the same *germ*.

The important object to study here are *derivations* of this algebra, meaning vector space homomorphisms

$$D\colon \mathcal{E}_{M,p} \to \mathbb{R},$$

satisfying the *Leibniz rule*

$$D(fg) = D(f)g(p) + f(p)D(g),$$

where $f(p), g(p)$ just stands for the evaluation at the point p of some representative of the germ. This clearly does not depend on the germ, and thus is well defined.

Now, the *tangent space* to the manifold M at the point p is the vector space of all derivations of the algebra $\mathcal{E}_{M,p}$. This tangent space is usually denoted by T_pM. One could naturally ask now what does this have to do with the tangent space that one can intuitively think about, for instance for a sphere in 3-dimensional Euclidean space. But if one looks carefully at this, one realizes that this abstract definition precisely reproduces the idea that one has intuitively. This is due to the fact that the algebra gives us a vector space of the same dimension as the manifold (for instance in two dimensions, just a plane).

To further convince oneself of this, one can verify that in fact

1. $\frac{\partial}{\partial x_j}$ for some local coordinates (x_j) are actually in fact derivations. The usual problem here is that we are so used to the natural identifications that we constantly use while doing calculus in $\mathbb{R}^n$, and one has to rethink those identifications, since they come from the translations of $\mathbb{R}^n$. But it is clear that in a general manifold, one does not have that structure, and thus this more abstract notion is needed.

2. $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ is a basis for the respective tangent space. Hence tangent vectors are just linear combinations of those.

The tangent bundle is constructed just by patching together this tangent spaces, meaning

$$TM = \dot{\bigcup}_{p\in M} T_pM,$$

where the dot stands for the fact that we are taking a disjoint union. This comes naturally equipped with a projection π taking every derivation to it's associated point. One can thus formally construct this bundle using this. We omit this construction, that is part of any book in differential geometry, and the interested reader is referred to [1] for more details.

Note that the same construction can be done in a complex manifold X by taking

$$\mathcal{O}_{X,x} = \varinjlim_{p\in U\subset M} \mathcal{O}_X(U),$$

which will be the $\mathbb{C}-$algebra of germs of holomorphic functions at $x \in X$. One constructs the *holomorphic (or complex) tangent space* in the same way as in the previous example, just by considering derivations over $\mathbb{C}$. In this setting, as before, complex coordinates induce derivations (and in fact a basis) over $\mathbb{C}$ as before. And in the same way one can take the union of all the tangent spaces to construct the tangent bundle in this complex setting. As before, we omit the concrete details of the construction, and the interested reader is referred to [1] for more details.

Given a bundle over a certain space, one can naturally restrict it to open sets, this motivates the following definition.

Definition 1.2.3. Let $\pi\colon E \to X$ be an $\mathcal{S}-$bundle and $U \subset X$ an open subset. Then we define the restriction of the bundle E to the open set U, which is denoted by $E|_U$, to be the $\mathcal{S}-$bundle

$$\pi|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to U.$$

Now that we have some notions about what a bundle is, as usual we want to know how to relate different bundles. To do this we introduce the natural notion of a morphism, that is just a morphism respecting all the structure a bundle has.

Definition 1.2.4. An $\mathcal{S}-$*bundle morphism* between two $\mathcal{S}-$bundles

$$\begin{aligned} \pi_E &\colon E \to X \\ \pi_F &\colon F \to X, \end{aligned}$$

is an $\mathcal{S}-$morphism $f\colon E \to F$ which takes fibres of E onto fibres of F. This naturally induces an $\mathcal{S}-$morphism $\hat{f}(\pi_E(e)) = \pi_F(f(e))$, which can be represented diagrammatically by the following (commutative) diagram:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ X \arrow[r, "\hat{f}"] & Y. \end{tikzcd}$$

One can in a natural way extend the algebraic (vector space) constructions that one has in linear algebra to the geometric setting of bundles by using the algebraic versions locally at every fiber. One can thus consider a wide variety of objects such as

- $A \oplus B$, the direct sum.
- $A \otimes B$, the tensor product.
- $Hom(A, B)$, the homomorphisms between A and B.
- A^*, the dual space.
- $\bigwedge^k A$, antisymmetric tensor products of degree k, usually called the *exterior algebra*.
- $S^k A$, symmetric tensor products of degree k, usually called the *symmetric algebra*.
- $\ker(f)$ for some bundle morphism f.
- $Im(f)$ for some bundle morphism f.

For more details on how to formally construct the bundles, the reader is referred to [1].

In the same manner as in linear algebra, one can introduce the notion of *exact sequences*, and thus try to build the whole machinery of *homological algebra* in this setting. Although this can be done and is useful, we will develop this machinery in the next chapter, using the theory of *sheaves*, which further generalize the notion of bundles. And thus we will later build a more general theory for developing homological algebra in a geometric setting, thus there is no need to introduce it here.

Vector bundles represent the geometry of the underlying base space. To get some understanding of these bundles, via analysis, one has to introduce the generalized notion of a function (which will reflect the geometry) where we will be able to apply the techniques of analysis.

Definition 1.2.5. Let $\pi\colon E \to X$ be an $\mathcal{S}-$bundle. Then an $\mathcal{S}-$morphism $\sigma\colon X \to E$ satisfying

$$\pi \circ \sigma = Id_X,$$

where $Id_X\colon X \to X$ is just the identity, is called an $\mathcal{S}-$*section* of the bundle.

Remark. This definition, although being very short, is fundamental in geometry and physics and its importance can not be overstated. One can see that what a section is doing is mapping points in the base space to their respective fibers, in an appropriate way.

The set of $\mathcal{S}-$sections of E over X is usually denoted by $\mathcal{S}(X, E)$. If we are interested in smaller domains, one can appropriately consider the restricted sections. Meaning that if $U \subset X$ then we denote the sections of $E|_U$ over U as $\mathcal{S}(U, E)$. There is also a common notation that we will sometimes use during the work. Provided that there is no confusion as to which category one is dealing with, the set of sections is usually denoted by $\Gamma(X, E)$.

Note that some very fundamental objects in differential geometry, for instance vector fields on a manifold, can be described by using sections. More concretely, vector fields are just sections of the tangent bundle over a manifold. This tells us that we can study vector fields on a manifold in the language of bundles and sections. In fact, almost all of classical differential-geometric objects of interest can be written in this language, differential forms, tensor fields...

One should note here that sections are in fact a generalization of vector-valued functions. For instance, if we have a differentiable manifold M, and we consider the trivial bundle $M \times \mathbb{R}$, then the set of sections is naturally identified with the smooth functions on M. And if one further considers higher rank trivial bundles, meaning $M \times \mathbb{R}^r$, then the sections are naturally identified with global differentiable mappings from M to $\mathbb{R}^r$. Due to the fact that vector bundles are locally like $U \times \mathbb{R}^r$, sections of this bundles are just vector valued functions locally, with the local representations appropriately related via the transition functions.

One should note that the space of sections $\mathcal{S}(X, E)$ for some vector bundle $\pi\colon E \to X$ is not merely a set. It can be equipped with some extra algebraic structures, due to the vector space structure of the fibers.

- **Vector Space:** Note that $\mathcal{S}(X,E)$ can be made into a $\mathbb{K}-$vector space. For two sections $\phi, \psi \in \mathcal{S}(X,E)$, adition is just defined in the natural way

$$(\phi+\psi)(x) \colon = \phi(x)+\psi(x).$$

  And scalar multiplication of a section $\phi \in \mathcal{S}(X,E)$ by $\alpha \in \mathbb{K}$ is just defined in the natural way

$$(\alpha\phi)(x) \colon = \alpha(\phi(x)).$$

- **Module:** $\mathcal{S}(X,E)$ can also be made into a module over the ring of functions $\mathcal{S}_X(X)$ by defining, for a section $\phi \in \mathcal{S}(X,E)$ and a function $f \in \mathcal{S}_X(X)$

$$(f\phi)(x) \colon = f(x)\phi(x).$$

  This perspective of viewing the space of sections as a module will be very useful in the next chapter when we deal with (pre)sheaves, and specially (pre)sheaves of modules.

Using the algebraic structures mentioned above, one can define certain bundles of interest. Here we let M be a differentiable manifold and TM its tangent bundle. In this setting one can define:

- The cotangent bundle T^*M.
- The exterior algebra bundles $\bigwedge TM$ and $\bigwedge T^*M$.
- The symmetric algebra bundles S^kTM and S^kT^*M.

With this bundles in our hand, one recovers notions such as differential forms, which are nothing more than sections of the exterior algebra bundles over a certain manifold. We will denote the space of differential forms of degree p over an open set U by either $\mathcal{E}^p(U)$ or $\Omega^p(U)$. This same procedure can be made for complex manifolds.

As usual in differential geometry, we will denote the dual basis of the basis of the tangent space $\{\frac{\partial}{\partial x_j}\}_j$ by $\{dx_j\}$.

On differential forms one has the usual exterior derivative operator

$$d \colon \mathcal{E}^p \to \mathcal{E}^{p+1},$$

which was introduced in its current form by the french mathematician Élie Cartan in [43].

In mathematics, the notion of *pullback* turns out to be very useful in a lot of areas. It allows one to pass certain structures from one space to another. We will like to introduce thus the notion of "pulling back" bundles. To do this we first need a technical result that justifies the later definition.

Lemma 1.2.1. *Given an $\mathcal{S}-$morphism $f\colon X \to Y$ and an $\mathcal{S}-$bundle $\pi\colon E \to Y$, then there exists an $\mathcal{S}-$bundle $\pi'\colon E' \to X$ and an $\mathcal{S}-$bundle morphism g making the following diagram commute*

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y. \end{array}$$

And moreover E' is unique up to equivalence.

For the proof of this fact, the reader is referred to [1] (Chap. I Proposition 2.15).

Definition 1.2.6. In the setting of the above lemma, the bundle E' is called the *pullback of E by f*, and is denoted by f^*E.

The notion of "pulling back" a bundle is very useful, although in this work we will by far not see it's full importance. One of the main used of the pullback bundle construction is in determining, up to equivalence, how many different vector bundles there are on a given space. The answer to the problem of determining the possible (isomorphism classes of) vector bundles depends on the type of manifold that one is considering. But for instance, in the case of differentiable manifolds, there is an important result that allows us to reduce the problem of bundle classification to a problem in homotopy theory. The result requires the use of the Grassmannian (real) manifold, its universal bundle and of the notion of homotopy between maps. If the reader is not comfortable with this notions, the result can be skipped, since it will not be fundamental for the later sections.

Theorem 1.2.1. *Let X be a differentiable vector bundle and $\pi\colon E \to X$ a differentiable vector bundle of rank r. Then there is a positive integer $N > 0$ (depending only on the base manifold X) and a differentiable mapping*

$$f\colon X \to G_{r,N}(\mathbb{R}),$$

where $G_{r,N}(\mathbb{R})$ is just the real Grassmannian manifold so that $f^*U_{r,N} \cong E$ (where $U_{r,N}$ is the universal bundle over the Grassmannian). Meaning that our original bundle can be seen (up to isomorphism) as the pullback of the universal bundle over $G_{r,N}$ via the map f.

Moreover, any other map g which is homotopic to f also satisfies $g^*U \cong E$.

Remark. The main conclusion that one extracts from the theorem is that one can classify (isomorphism classes of) vector bundles over X just by studying the homotopy classes of maps from the base space X into the Grassmannian. For certain spaces, this homotopy classes turn out to be computable (see for instance [49]) this is the case for instance for X being the sphere.

If one furthermore has that X is compact, then one can in fact require the map f to be an *embedding*, although it may be necessary to consider a bigger N in that case. Note that, due to the homotopy property, the theorem is actually also valid in the more general setting of continuous vector bundles (not requiring smoothness). And in fact there will be a one-to-one correspondence between (isomorphism classes of) continuous and differentiable (and also real-analytic) vector bundles. Hence in the real case, one has reduced the problem to classifying the type of bundle that one prefers, continuous, differentiable or analytic.

The important thing to note here, and this is one of the first places where we start to see that complex and real geometry are in fact very different, is that the same result is **not true** in the case of holomorphic vector bundles over a compact (complex) manifold. One has to impose some additional *positivity conditions*.

In fact, we will in later sections deal with the problem of embedding compact complex manifolds into projective space, and we will reduce the problem, following [1], to finding a certain class of holomorphic bundles so that a certain analog of the theorem above holds, and such that the map f gives thus an embedding into the complex Grassmannian.

One of the fundamental references for understanding this kind of classification problems is the classical book by Steenrod [49]. Since we will not need this concrete result later on, we refer the interested reader to Steenrod's book, where this questions are explained.

Here we would like to briefly comment on some ideas that we believe are related to this bundle pullback construction and specially to Grassmannians, and that are being researched by some theoretical physicists nowadays with great interest.

To understand what does this have anything to do with physics, we have to take a brief look at the history of physics in the second half of last century. At first sight, projective spaces (the idea that Grassmannians generalize) seem to have nothing to do with physics. But that is very far from the truth if one looks at it carefully, and in fact it has become an indispenasble tool for modern theoretical physics. One of the main proponents of introducing techniques that use the projective space into physics has been the british mathematical physicist Roger Penrose.

Some of the ideas that motivated Penrose were the fact that one could get some solutions to equations of physical interest (massless fields of arbitrary spin) by integrating certain holomorphic objects over submanifolds of projective space. The interested reader is referred to a paper by Penrose on the topic, namely [29], where the explicit way to do this and construct the solutions is explained.

With this type of ideas in mind, Penrose formulated a revolutionary idea, and that became known as *twistor theory*. We will not even attempt to cover the theory, and the interested reader is referred to the books that Penrose wrote on the topic, namely [30] for instance. Penrose saw his theory as a way to deal with the question of Quantum Gravity, and although the problem of quantizing gravity is far from being understood still nowadays, this ideas have not left the minds of theoretical physicists since the 70's.

Once the ideas were starting to become more known, a lot of physicists and mathematicians, such as Ward, Atiyah, Hitchin... started to use this ideas to deal with problems both in physics and mathematics. And many physical notions started to become understood in terms of *twistors*. But how does this theory have anything to do with pullbacks and Grassmannians?

As we said, Grassmannians are a generalization of Projective spaces, and thus one could think that, if the theory of twistors (formulated in projective space) is useful for physics, why not Grassmannians? This was the motivation for a group of theoretical physicists led by Nima Arkani-Hamed and Jaroslav Trnka to propose that the Grassmannian (in fact a positive Grassmannian) can be used to compute certain quantities (scattering amplitudes) which are of fundamental importance for understanding the inner workings of nature at a small scale. They formulated their ideas in terms of a geometric object that they call the *Amplituhedron*.

This is still today an active research topic, and the physical community is far from understanding if this ideas can be applied to get a unified description of the forces of nature, and quoting Edward Witten (one of the most important string-theorists), who said that the results were very unexpected

It is difficult to guess what will happen or what the lessons will turn out to be.

What seems clear, is that there are many open paths in today's Particle Physics research, and they could turn out not to describe our universe. What we would like to remark is that we believe that mathematicians should look closer at this ideas, because many great mathematical theories have grown out of physics.

And finally, from the perspective of a geometer, the important thing to note is that their ideas imply that the whole of physics (at a fundamental level) can be **formulated in terms of geometry**. And as was a revolution to see, in the beginning of last century, that gravity could be described in terms of differential-geometric notions, this could be a revolution having similar implications, for both physics and mathematics.

Of course one should note that these theories are **highly speculative** as understood nowadays, and that a lot of research remains to be done, since this ideas only work nowadays with what physicists call *toy models*. The reader interested in this ideas is referred to the following papers, that introduced the ideas and certain computations using them into physics, namely [56] and [57].

1.3 Almost-complex structures and associated operators

In this section, we will introduce some first-order differential operators acting on differential forms that will intrinsically reflect the complex structure. The natural context in which to introduce this operators is from the viewpoint of so-called *almost-complex* manifolds. These structures are generalizations of complex structures having the same structure of a complex manifold but only at the first order (meaning at the tangent space level). We will start by discussing the notion of a $\mathbb{C}-$linear structure on a real vector space. We will later apply this notions coming from linear algebra to geometry, as is usual in geometry, where we usually start by understanding the proper algebraic construction to later "put them in motion" in a geometric setting. In a sense one could say that differential geometry is linear algebra in motion, and much more, of course (since not everything in geometry is reduced to first order).

We begin with a definition that will be fundamental for introducing complex structures in real vector spaces.

Definition 1.3.1. Let V be a real vector space, and suppose that we have an $\mathbb{R}-$linear isomorphism

$$J\colon V \to V,$$

satisfying the fundamental property $J^2 = -Id_V$. If J satisfies this conditions, it is called a *complex structure* on the vector space V.

Remark. The fundamental reason that motivated this definition is the fact that, given a real vector space V and a complex structure J on V, then we can equip V with the structure of a complex vector space. To do this we define the complex scalar multiplication as:

$$(a+ib)v\colon = av + bJv,$$

for $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. And thus with this definition we have scalar multiplication on V by complex numbers, and equipped with this scalar multiplication V becomes a complex vector space.

Going in the converse direction, if V is a complex vector space, then clearly it can be considered as a vector space over the real numbers. Multiplication by i in this space is an $\mathbb{R}-$linear endomorphism of V that we can call J, and it is clearly a complex structure. It is in fact this what the definition generalizes. Moreover, if $\{v_1, ..., v_n\}$ is a basis for V over the complex numbers, then $\{v_1, ..., v_n, Jv_1, ..., Jv_n\}$ is a basis for V as a real vector space.

Now, consider that V is a real vector space with a complex structure, and consider $V \otimes_{\mathbb{R}} \mathbb{C}$ to be the complexification of V. We can extend the $\mathbb{R}-$linear map J to a $\mathbb{C}-$linear mapping on the complexification by defining $J(v \otimes \alpha) = J(v) \otimes \alpha$ for every $v \in V, \alpha \in \mathbb{C}$. Moreover, this extension that we defined clearly also satisfies the property that $J^2 = -Id$, and thus J has two eigenvalues $\{i, -i\}$. We denote by $V^{1,0}$ the eigenspace corresponding to the eigenvalue i and by $V^{0,1}$ the eigenspace associated with $-i$. Then we have the splitting of the complexification into a direct sum as follows:

$$V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}.$$

We also have another operation in this space, namely conjugation, which is defined by $\overline{v \otimes \alpha} = v \otimes \overline{\alpha}$ for every $v \in V, \alpha \in \mathbb{C}$. Conjugation induces an isomorphism $V^{1,0} \cong_{\mathbb{R}} V^{0,1}$, and conjugation is a conjugate-linear mapping. One can see that if one considers the complex vector space which is obtained from V by means of the complex structure J as we defined before is $\mathbb{C}-$linearly isomorphic to $V^{1,0}$, and we will usually identify both spaces from now on.

Exterior algebras are a fundamental tool in differential geometry, and in this (almost)-complex setting they are very useful too. We denote the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ by V_c and consider the exterior algebras

$$\bigwedge V_c \ ; \ \bigwedge V^{1,0} \ ; \ \bigwedge V^{0,1}.$$

In this spaces we have the natural injections

$$\bigwedge V^{1,0} \to \bigwedge V_c$$
$$\bigwedge V^{0,1} \to \bigwedge V_c$$

in this exterior algebras, of specially importance will be the subspaces $\bigwedge^{p,q} V \subset \bigwedge V_c$ which are generated by elements of the form $u \wedge v$ with $u \in \bigwedge^p V^{1,0}$ and $v \in \bigwedge^q V^{0,1}$. With this subspaces in our hands, and if we denote by $n = dim_{\mathbb{C}} V^{1,0}$ then we have the direct sum decomposition

$$\bigwedge V_c = \bigoplus_{r=0}^{2n} \bigoplus_{p+q=r} \bigwedge^{p,q} V.$$

Thus what we see is that in this setting, the exterior algebra gets a richer structure, that will be very useful in the study of complex geometry. We now want to put this notions "in motion" as we said before, and introduce the appropriate definitions for applying this linear algebra constructions into geometry.

Definition 1.3.2. If X is a differentiable manifold of even dimension $2n$, and J is a differentiable vector bundle isomorphism

$$J \colon T(X) \to T(X),$$

such that for every point $x \in X$ the map $J_x \colon T_xX \to T_xX$ is a complex structure for the real vector space T_xX, meaning that $J^2 = -I$, where I is the identity vector bundle isomorphism acting on the tangent bundle $T(X)$.

J is called an *almost complex structure* for the differentiable manifold X. If the manifold X is equipped with such structure, it is called an *almost complex manifold.*

One can easily see that the name is for a good reason, since it is a generalization of the notion of a complex manifold, in the sense of the following fact.

Lemma 1.3.1. *A complex manifold X induces an almost complex structure on its underlying differentiable manifold.*

This proof reduces to checking that the complex structure that we have at every fiber T_xX_0, where X_0 is the underlying differentiable manifold of X, satisfies the desired requirements. We omit the proof and the interested reader is referred to [1] (Chap. I Proposition 3.4).

Now we want to introduce the aforementioned exterior algebra constructions into this geometric setting. If we have a differentiable $m-$manifold, and we consider the complexified tangent bundle $T(X)_c = T(X) \otimes_{\mathbb{R}} \mathbb{C}$ and respectively the complexified cotangent bundle, that we denote by $T^*(X)_c$. By applying the linear algebra constructions, we can form the exterior algebra bundle $\bigwedge T^*(X)_c$, and we let

$$\mathcal{E}^r(X)_c = \mathcal{E}(X, \bigwedge^r T^*(X)_c,$$

these objects are called *complex-valued differential forms of total degree* r on X. To ease notation, they are usually denoted by $\mathcal{E}^r(X)$ (by dropping the subscript c) whenever there is no confusion between these and real-valued forms. If we consider local coordinates, we have a (complex) form $\alpha \in \mathcal{E}^r$ if and only if α can be expressed in a coordinate neighborhood as $\alpha(x) = \sum'_{|I|=r} \alpha_I(x) dx_I$, where $\alpha_I(x)$ are just $\mathcal{C}^\infty$ complex values functions on the neighborhood. Note that we are using the usual multiindex notation, and $\sum'$ just means that indices are taken to be strictly increasing.

One knows from real differential geometry that the exterior derivative d plays a fundamental role (as was emphasized by Élie Cartan), and one is thus encouraged to extend the notion to complex geometry. This is easy since the operator d can be extended by imposing complex linearity to act on complex-valued differential forms, thus inducing the sequence

$$\mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^m(X) \to 0,$$

where the fundamental property $d^2 = 0$ is clearly still satisfied.

If (X, J) is an almost-complex manifold, then we can apply the linear algebra constructions that we defined before to the complexified tangent bundle TX_c. The map J extends naturally to a $\mathbb{C}-$linear isomorphism on TX_c having (fiberwise) eigenvalues $+i$ and $-i$. As before, we denote by $TX^{1,0}$ the bundle of $+i$ eigenspaces and $TX^{0,1}$ the bundle of $-i$ eigenspaces for the map J (note that those are differentiable subbundles of TX_c).

One can thus also extend the notion of conjugation to TX_c

$$Q\colon TX_c \to TX_c,$$

by just using the fiberwise conjugation. As before we have that

$$Q\colon TX^{1,0} \to TX^{0,1}$$

is a conjugate-linear isomorphism. We denote by $T^*X^{1,0}$ and $T^*X^{0,1}$ the $\mathbb{C}-$dual bundles associated to $TX^{1,0}$ and $TX^{0,1}$ respectively. We have that

$$T^*X_c = T^*X^{1,0} \oplus T^*X^{0,1}.$$

We can considerate the associated exterior algebras for each of the bundles, and we end up with natural inclusions

$$\bigwedge T^*X^{1,0} \to \bigwedge T^*X_c,$$

and
$$\bigwedge T^*X^{0,1} \to \bigwedge T^*X_c.$$

We will denote by $\bigwedge^{p,q} T^*X$ the bundle whose fibre is $\bigwedge^{p,q} T^*_x X$ for every point x. This bundle is of fundamental importance for complex geometry, since its sections are the so-called *complex-valued differential gorms of type* (p,q), that we denote by

$$\mathcal{E}^{p,q}(X) = \mathcal{E}(X, \wedge^{p,q} T^*X),$$

moreover we note that we have a decomposition

$$\mathcal{E}^r(X) = \bigoplus_{p+q=r} \mathcal{E}^{p,q}(X).$$

One should note that, while differential forms of degree r do not reflect the underlying complex structure J that we have in our manifold, the decomposition into subspaces (p,q) does depend on it clearly (by construction).

To compute with differential forms of this kind, as with the more usual differential forms, we use local coordinates (or *frames*, that will be introduced in a later chapter). Any section $\omega \in \mathcal{E}^{p,q}(X)$ can be written locally as $\omega = \sum'_{|I|=p,|J|=q} \omega_{IJ} dz^I \wedge d\overline{z}^J$, in local coordinates z_j.

Note that the exterior derivative d in this setting acts on ω (in local coordinates) as $d\omega = \sum'_{|I|=p,|J|=q} d\omega_{IJ} \wedge dz^I \wedge d\overline{z}^J + \omega_{IJ} d(dz^I \wedge d\overline{z}^J)$. Clearly the second term is just zero in this setting, due to the fundamental property of the exterior derivative. We write it since for more general *non-holomorphic frames* it may not vanish, and one should be aware of that fact, although during this work we will mainly be in the setting where this term just vanishes.

The decomposition of differential forms into the bidegree (p,q) has some associated projection operators
$$\pi_{p,q} \colon \mathcal{E}^r \to \mathcal{E}^{p,q}(X), \quad p+q=r.$$

In this setting,
$$d \colon \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+q+1} = \bigoplus_{r+s=p+q+1} \mathcal{E}^{r,s}(X),$$

if we restrict d to $\mathcal{E}^{p,q}$. Endowed with this objects, we define two operators of fundamental importance for complex geometry, the so-called *Dolbeaut operators*. This operators were introduced by the french mathematician Pierre Dolbeaut (who was a student of Henri Cartan) in his doctoral dissertation [55] generalizing the *Wirtinger derivatives* from multivariate complex analysis. In his dissertation, Dolbeaut introduced many of the notions and results that we will develop during this work, and was a fundamental contributor to the development of complex analytic geometry.

We define the operators

$$\partial\colon \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q}(X)$$
$$\overline{\partial}\colon \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q+1}(X)$$

by

$$\partial = \pi_{p+1,q} \circ d$$
$$\overline{\partial} = \pi_{p,q+1} \circ d.$$

And we then extend both operators ∂ and $\overline{\partial}$ to all

$$\mathcal{E}^*(X) = \bigoplus_{r=0}^{\dim X} \mathcal{E}^r(X),$$

just by complex linearity.

The relation between both operators is via the conjugation operator Q, and is just given by

$$Q\overline{\partial}(Q\omega) = \partial\omega,$$

for an arbitrary $\omega \in \mathcal{E}^*(X)$. This is given by (Chap I. Proposition 3.6) in [1]. And has a very simple proof just by using the definitions of each operator for every (p, q), and the fact that conjugation commutes with the exterior derivative in this setting.

One should note that, for an arbitrary J (almost-complex structure), although we have that $d^2 = 0$, it is not necessarily the case that that the Dolbeaut operators also square to zero. From the relationship above we can deduce that

$$\partial^2 = 0 \iff \overline{\partial}^2 = 0,$$

but this doesn't happen for every J, as we said.

The exterior derivative d, in this setting can be decomposed by using the direct sum decomposition as

$$d\colon \mathcal{E}^{p,q} \to \mathcal{E}^{p+q+1}$$
$$d = \sum_{r+s=p+q+1} \pi_{r,s} \circ d = \partial + \overline{\partial} + \cdots .$$

Thus in general there are a lot of operators in this decomposition of d. However, if for a certain almost-complex manifold (X, J) we have that $d = \partial + \overline{\partial}$, then we deduce that

$$d^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2 = 0,$$

and thus, since by definition every operator projects to a different summand of the image $\mathcal{E}^{p+q+2}(X)$, we obtain from the previous equation three things:

$$\partial^2 = 0$$

$$\overline{\partial}^2 = 0$$
$$\partial\overline{\partial} + \overline{\partial}\partial = 0,$$

and thus we have a lot more of structure, as we will see in later sections. This important case has a definition due to its importance.

Definition 1.3.3. If for an almost-complex manifold (X, J) we have that $d = \partial + \overline{\partial}$, then we say that the almost-complex structure J is *integrable.*

But now the natural question to ask is, when is an almost-complex structure integrable? Is the almost-complex structure associated with a complex manifold integrable? Both questions have an answer, and we thus have a complete characterization.

We start with the second question, since in a sense it is intuitive that a complex manifold should have a "very good" almost-complex structure. At a naive level, we are removing the word "almost" from the expresion, does that mean something? And indeed it does.

Theorem 1.3.1. *If X is a complex manifold, then its associated almost-complex structure is integrable.*

Proof. We denote by (X_0, J) the underlying differentiable manifold with the induced almost-complex structure J coming from X. As $\mathbb{C}-$bundles, we have that

$$TX \cong TX_0^{1,0},$$

and similarly for the dual bundles

$$T^*X \cong T^*X_0^{1,0}.$$

If $(z_1, ..., z_n)$ are local complex coordinates, then those induce a basis for each of the tangent and cotangent spaces at every point, namely $\{\frac{\partial}{\partial z_j}\}$ and $\{dz_j\}$. With this we define an associated local basis $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\}$ for $T(X_0)_c$ as follows (note this is just the definition of the usual Wirtinger Derivatives)

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$
$$\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$

Note that this just defines a change of basis. From this definition one deduces that

$$dz_j = dx_j + idy_j$$
$$d\overline{z}_j = dx_j - id_j$$

just by applying the definitions. And from this we get

$$dx_j = \frac{1}{2}(dz_j + d\overline{z}_j)$$
$$dy_j = \frac{1}{2i}(dz_j - d\overline{z}_j).$$

Now, for any differential form $\alpha = \sum'_{I,J} \alpha_{IJ} dz^I \wedge d\overline{z}^J \in \mathcal{E}^{p,q}(X)$, the exterior derivative is

$$\begin{aligned} d\alpha &= \sum_{j=1}^{n} \sum_{IJ}' \left(\frac{\partial \alpha_{IJ}}{\partial x_j} dx_j + \frac{\partial \alpha_{IJ}}{\partial y_j} dy_j\right) \wedge dz^I \wedge d\overline{z}^J \\ &= \sum_{j=1}^{n} \sum_{IJ}' \frac{\partial \alpha_{IJ}}{\partial z_j} dz_j \wedge dz^I \wedge d\overline{z}^J + \sum_{j=1}^{n} \sum_{IJ}' \frac{\partial \alpha_{IJ}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz^I \wedge d\overline{z}^J. \end{aligned}$$

Note that this is just a sum of a $(p+1, q)$ term and a $(p, q+1)$ term. Thus we have seen that

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j,$$

and

$$\overline{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \overline{z}_j} d\overline{z}_j,$$

and thus the above computation proves that $d = \partial + \overline{\partial}$, as we wanted. Hence the almost-complex structure induced by the complex manifold structure of X is integrable, and this concludes the proof.

□

Recall that we had two questions. One of them is answered, the almost-complex structure of a complex manifold is integrable, justifying the terminology. The other question turns out to be far harder to answer, giving a complete characterization of integrable almost-complex structures is a non-trivial problem.

The solution to this question turns out to be a theorem, which is known in the literature as the *Newlander-Nirenberg* theorem [28]. The theorem solves our question, and in fact tells us that an almost-complex manifold (X, J) with J being integrable in fact can be equipped with a unique complex structure $\mathcal{O}_X$ turning X into a complex manifold with induced almost-complex structure J.

We will not prove this result, and the reader is referred to the original proof by Newlander and Nirenberg in [28], or alternatively to the proof which is in the book [27] by one of the best complex analyst of all times, the swedish Lars Hörmander. In the book the problem is reduced to solving a certain PDE with some estimates, a technique that Hörmander applied with great success to a wide variety of problems.

We will end the chapter by noting that there is an alternative vision on the integrability of an almost-complex structure which is equivalent to the one we gave. To introduce this notion we have to introduce a certain tensor field, which is called the *Nijenhuis tensor* N_J, and depends on the almost-complex structure J. This tensor is defined, in terms of the Lie Bracket of vector fields, as

$$N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y].$$

And it is precisely a measure of the failure of J to be integrable, since J will be integrable if and only if this tensor vanishes everywhere.

2 Sheaf theory

In this chapter we will introduce the theory of *sheaves on topological spaces.* Sheaves are a fundamental tool in many areas of mathematics, due to their ability to deal with the problem of relating local and global information. In the case of (complex) manifolds, this is very useful, since we do calculus locally (and physics) but we also need to take into account the global properties. Gluing together "solutions" to certain problems in a coherent manner to describe, for instance, global invariants, is easily accomplished using the language of sheaves, and their cohomological theory.

The power of a sheaf-theoretic perspective is not in providing the unique way to solve a problem, since usually most problems can be phrased (and even solved) without even using the word sheaf. But if one did that, one would loose the abstract perspective that sheaves give us, and many proofs would be much more complicated and difficult to understand. Hence the usefulness of sheaves lies in being precisely the right abstraction to deal with a wide variety of problems.

Before we start with the mathematical theories, we believe that the man that contributed to introduce them deserves a mention. And the history behind the development of sheaf theory is very beautiful but also full of pain, and we believe that it can show us many things about what mathematics should be, a critical tool for understanding the world around us. Mathematics is not a neutral subject, as one could think, and the man who invented sheaves had to choose. Either he used mathematics for developing oppression, or for fighting for freedom, and his courage and political views made him choose the second path. But what does this have to do with sheaves?

Sheaves were invented by the french mathematician Jean Leray (he was also an officer of the french army), who was an expert in PDE's when he started to develop the theory. In fact, Leray's PhD thesis was about weak solutions to the Navier-Stokes equations [58], something that a priori is not very related to the usual algebro-topological/algebro-geometric setting where sheaves are usually introduced nowadays. And although Leray used some ideas from algebraic topology for his results in Fluid Dynamics, he was by far not an expert on the field, as he said. But now, what made Leray change its research and devote to the study of abstact topological spaces? This is a bit of a sad story, but at the same time full of dignity, quoting the swiss mathematician Armand Borel in [74]:

The Second World War broke out in 1939 and J. Leray was made prisoner by the Germans in 1940. He spent the next five years in captivity in an officers' camp, Oflag XVIIA in Austria. With the help of some colleagues, he founded a university there, of which he became the Director ("recteur"). His major mathematical interests had been so far in analysis, on a variety of problems which, though theoretical, had their origins in, and potential applications to, technical problems in mechanics or fluid dynamics. Algebraic topology had been only a minor interest, geared to applications to analysis. Leray feared that if his competence as a "mechanic" ("mécanicien", his word) were known to the German authorities in the camp, he might be compelled to work for the German war machine, so he converted his minor interest to his major one, in fact to his essentially unique one, presented himself as a pure mathematician and devoted himself mainly to algebraic topology.

We believe that this is a great lesson in political compromise, and could not be overstated. In a world where mathematics (along with the sciences) was used as a war tool for many governments around the world, Leray decided that he preferred to risk his life than to let the Nazis know he knew anything about mechanics. And thus he devoted to study "abstract nonsense" for the Nazi officials.

In a world where mathematically-based algorithms are becoming the rule, we believe that the role of mathematicians is to demystify this algorithms. To apply critical reasoning, and don't let governments and corporations use mathematics for reproducing oppression. If we are not careful, the same thing that happened to many physicists during past century can happen to mathematicians this century. There is a famous quote by the american physicist J. Robert Oppenheimer, who reproduced the words of the Hindu script *Bhagavad Gita*:

I am become Death, the destroyer of worlds.

Referring to the nuclear bombs that the american government used against the civil population, and that his nuclear research helped develop. We think that this should be an important debate.

Now, let's dive into the mathematical world that Leray started to invent in the *séminaire* at the concentration camp.

2.1 Presheaves and Sheaves

In this section we will devote to the formal study of *presheaves* and *sheaves* and motivate the notions with examples, and those are abundant since this notions abstract away a situation that is common to a lot of situations in mathematics and specially in geometry. We will try also to give a more abstract description using the language of category theory whenever possible.

Definition 2.1.1. A *presheaf* $\mathcal{F}$ over a topological space X is

- An assignment of a set $\mathcal{F}(U)$ to each open set $\mathcal{U} \subset X$.
- A collection of maps (called *restriction homomorphisms*)

$$r^U_V \colon \mathcal{F}(U) \to \mathcal{F}(V)$$

  for every pair of open sets $V \subset U \subset X$. If U, V, W are open sets, this maps have to satisfy:

  - $r^U_U \colon \mathcal{F}(U) \to \mathcal{F}(U)$ is just the identity.
  - If $W \subset V \subset U$ then $r^U_W = r^V_W \circ r^U_V$.

Remark.

- The maps r^U_V are called restriction maps since the main examples that we will later see consist of functions (of several classes) and this maps will just be the usual restriction of functions. And we will sometimes denote the restriction mappings $r^U_V \colon \mathcal{F}(U) \to \mathcal{F}(V)$ just by $f \to f \restriction_V$.
- From a categorical viewpoint, the definition of a presheaf is much shorter and clearer. Presheaves turn out to be nothing more than contravariant functors from one category to another. The first category at play comes from the topological data that we have in our topological space X. From its topology we can construct the *category of open subsets* of the topological space, that we denote by $Open(X)$. This category has open subsets of X as objects and the inclusion of open sets $\iota \colon V \hookrightarrow U$ as maps in the category.

  Now we can see a presheaf $\mathcal{F}$ over X just as a **contravariant functor** from the category $Open(X)$ to the category of sets that we denote by Set:

$$\mathcal{F} \colon Open(X) \to Set.$$

  If one unpacks the notions we just end up with the same definition as before, thus two equivalent ways to think about presheaves. Now one can see that the Set category was a bit arbitrary and we could have constructed functors into other categories, such as Abelian Groups or R-modules (the main examples of interest for us in doing geometry). And we will later study in more detail (pre)sheaves that have some aditional structure for every open set, that furthermore are required to induce the same structure in all it's substructures and related maps (that is imposed by the definition as a functor onto another catagory $\mathcal{C}$).

Now we will connect to the previous notions developed introduced in differential geometry and will give a special name (on purpose) to the elements of our sheaf, that reminds us of *sections of bundles.*

Definition 2.1.2. Let $\mathcal{F}$ be a presheaf over a topological space X. The elements $f \in \mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ over U.

Remark. The exact reason why we call these elements sections will become much more apparent later when we start seeing examples, but for the moment think about a bundle E over a manifold M. The space of sections $\Gamma(E)$ will turn out to be a presheaf, thus seeing that the sheaf-theoretic notions also encompass the notions of more classical differential geometry.

Now that we have the notion of (pre)sheaf, we can define, in the most *natural* way the *morphisms* between them.

Definition 2.1.3. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves over a topological space X then we define a *morphism* (of presheaves) as a collection of maps

$$\eta_U \colon \mathcal{F}(U) \to \mathcal{F}(U)$$

for every open set U of our topological space X. The maps have to make the following diagram commute

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "r_V^U"'] & \mathcal{G}(U) \arrow[d, "r_V^U"] \\
\mathcal{F}(V) \arrow[r, "\eta_V"] & \mathcal{G}(V)
\end{tikzcd}$$

for every open sets $V \subset U$ in X.

Remark.

- If the category being the codomain of the presheaf is different from the category of sets the morphisms η_U are required to be morphisms in the appropriate category (e.g. group homomorphisms, ring homomorphisms...).
- We call $\mathcal{F}$ a *subpresheaf* of $\mathcal{G}$ if all the maps η_U defining the morphism are inclusions. As before, subpresheafs are required to have the induced substructure in the appropriate category.
- We can also see these morphisms from a more categorical point of view, and they turn out to be a fundamental object of study in category theory, *natural transformations.* In fact one could say, following the american mathematician Saunders Mac Lane, one of the fathers of category theory, that the notion of natural transformation was in fact what motivated the theory, even more than functors. Quoting Mac Lane in

[77]: "*I didn't invent categories to study functors; I invented them to study natural transformations.*"

Given two presheaves $\mathcal{F}, \mathcal{G}$ over our topological space X, recall that they can be seen as two contravariant functors

$$\mathcal{F} \colon Open(X) \to Set$$
$$\mathcal{G} \colon Open(X) \to Set$$

and a morphism between these as presheaves is just a natural transformation between the associated functors

$$\eta \colon \mathcal{F} \implies \mathcal{G}.$$

This together with the remark on the definition of presheaf allow us to conclude that the study of presheaves is just the study of contravariant functors from the open category onto the category of sets (or others) and their natural transformations, this abstract perspective enables us to have an abstract view on a lot of situations that are common in a lot of geometric settings. This abstract view was pursued by the french mathematician Alexander Grothendieck in his revolutionarization of algebraic geometry, rewriting it entirely using the abstract notions of categories, functors, natural transformations, (pre)sheaves...

Now that we have a firm grip of what presheaves are we want to define a refinement of the notion, including some properties that will be of use in geometric situations. With the following definition it will come clear why the prefix "pre" has been necessary, since we are ready now to define the notion of *sheaf*.

Definition 2.1.4. Let $\mathcal{F}$ be a presheaf. We say that $\mathcal{F}$ is a *sheaf* if furthermore, for every collection of open sets $\{U_i\}$ with $U = \cup_i U_i$ our presheaf $\mathcal{F}$ satisfies two more properties

- If $\tau, \sigma \in \mathcal{F}(U)$ and they agree on every restriction, meaning

$$r^U_{U_i}(\tau) = r^U_{U_i}(\sigma)$$

for every i then we must have that $\tau = \sigma$ in $\mathcal{F}(U)$. (*If two sections agree on every set of an open cover then they coincide.*)

- If we have a family of local sections $\sigma_i \in \mathcal{F}(U_i)$ and if for $U_i \cap U_j$ non-empty we have

$$r^{U_i}_{U_i \cap U_j}(\sigma_i) = r^{U_j}_{U_i \cap U_j}(\sigma_j)$$

for every i, then there has to exist a section $\sigma \in \mathcal{F}(U)$ satisfying

$$r^U_{U_i}(\sigma) = \sigma_i.$$

(*Local sections can be glued together if certain compatibility conditions are met and form a section in a bigger open set*).

Remark.
- The notion of *morphism* of sheaves is simply a morphism of the underlying presheaf.
- If we have $\mathcal{G}$ a subpresheaf of a sheaf $\mathcal{F}$ and $\mathcal{G}$ is also a sheaf, $\mathcal{G}$ is called a *subsheaf.*
- An isomorphism of sheaves (or presheaves) $\eta\colon \mathcal{F} \implies \mathcal{G}$ is defined just by requiring that the maps η_U are isomosphisms in the category under consideration for each $U \in Open(X)$.

Now that we have the abstract definitions and machinery that will be used during the chapter, it is now time to see examples of sheaves. And this will be fundamental to realize the fact that (pre)sheaves are just the necessary level of abstraction to properly deal with many geometric situations. One should note that getting the appropriate degree of abstraction is not at all an easy task, and can be very tricky, but at the same time is a fundamental task for understanding mathematical theories and many other fields. The dutch theoretical computer scientist Edsger W. Dijkstra, known in the mathematical community for his work in graph theory, said that

The purpose of abstraction is not to be vague, but to create a new semantic level in which one can be absolutely precise.

In a sense, the "bird view" that abstraction can provide, and that made Grothendieck's work so important for modern mathematics, can turn into a view that forgets fundamental concrete details if one is not careful. The american mathematician John Tate, who has made fundamental contributions to a lot of fields of mathematics, commented some years ago in an interview about the power that finding the right abstractions can have. When asked about Grothendieck's work he said in [13]:

He just had an instinct for the right degree of generality. Some people make things too general, and they're not of any use. But he just had an instinct to put whatever theory he thought about in the most general setting that was still useful. Not generalization for generalization's sake but the right generalization. He was unbelievable.

Abstraction is a fundamental tool for understanding the world, but can be a drawback if one uses its power badly. There is a well known phrase that was used in a famous film but that comes in fact from the times of the french revolution, and we believe that applies here in an essential way:

Une grande responsabilité est la suite inséparable d'un grand pouvoir. [45]
(With Great Power Comes Great Responsibility)

Now let's look at some concrete examples.

Example 2.1. If we consider two topological spaces $X, Y \in Top$, we consider the presheaf $\mathcal{C}(X, Y)$ over X defined by

- $\mathcal{C}(X,Y)(U) = \{f : U \to Y continuous\}$.
- For every section $f \in \mathcal{C}(X,Y)(U)$ the restriction maps $r_V^U(f) = f \restriction_V$ is just the usual restriction of functions.

One quickly checks that $\mathcal{C}(X,Y)$ satisfies the axioms of a presheaf over the topological space X (in fact the notion of presheaf is constructed to generalize objects such as $\mathcal{C}(X,Y)$).

Furthermore, if we have an open covering $U = \cup_i U_i$ and two sections $f, g \in \mathcal{C}(X,Y)(U)$, these are just continuous functions

$$f : U \to Y$$
$$g : U \to Y$$

that satisfy,

$$f \restriction_{U_i} = g \restriction_{U_i}$$

for every i then clearly the functions f, g are the same. Thus we see that the presheaf $\mathcal{C}(X,Y)$ satisfies the first axiom of a sheaf. And as far as the second axiom goes, if we have a family $f^i \in \mathcal{C}(X,Y)(U_i)$ and for $U_i \cap U_j \neq$ we have

$$f^i_{U_i \cap U_j} = f^j_{U_i \cap U_j}$$

for every i, then we should be able to patch these local sections appropriately. By constructing the continuous function $f \in \mathcal{C}(X,Y)(U)$ in the open set $U = \cup_i U_i$

$$f : U \to Y$$
$$f(p) = f^i(p) \quad , p \in U_i$$

just by gluing together the local continuous functions. This construction gives us a well defined function since the local functions f^i are required to agree on the intersections, and thus f is a well defined continuous function in the open set U.

Hence we have seen that $\mathcal{C}(X,Y)$ is actually a **sheaf** over the topological space X. We will call this the *sheaf of continuous functions* from X to Y. And is one of the fundamental examples that motivates the definition of a sheaf. Many of the other sheaves that appear in geometry are actually sub(pre)sheafs of this bigger sheaf, since continuity will always be present in our study as a requirement (and usually we will work in the smooth setting).

Now that we have this big family of examples of sheaves we can look at particular subsheafs that may be of interest since they may allow us to introduce further algebraic data in our (pre)sheaf.

Example 2.2. Let X be a topological space, and denote by $\mathbb{K}$ the ground field of interest (usually for us it will always be either $\mathbb{R}$ or $\mathbb{C}$). And now we consider the sheaf of continuous functions from our topological space to the ground field viewed as a topological space

$$\mathcal{C}_X = \mathcal{C}(X, \mathbb{K})$$

And now more than being merely a sheaf into the category Set, we do have more algebraic structure in this sheaf. This sheaf is actually a functor

$$\mathcal{C}_X : Open(X) \to \mathbb{K} - Alg$$

from the category of open sets of our topological space X to the category of $\mathbb{K}-$algebras since for every open set $U \subset X$ the image object $\mathcal{C}_X(U)$ forms an algebra over the field $\mathbb{K}$ defined by pointwise addition, multiplication and multiplication of functions by scalars.

So, we have now seen our first example of a sheaf that has some algebraic information attached, this is going to be an important idea during this section and the next ones, since the algebraic structure attached to the sheaf is very important for understanding the geometric information we can get out of it.

Now, in a lot of geometric situations, appart from the notion of continuity there are other analytical notions that restrict the class of functions under consideration. To allow us to make a deeper study of their properties, we can require functions to be of class $\mathcal{C}^k$ (k-times differentiable), smooth (we mean by this $\mathcal{C}^\infty$ functions), real analytic functions, holomorphic functions (in the complex setting) and much more... These objects that appear naturally in geometric situations can also be described with the language of sheaves.

Example 2.3. If we consider X to have more structure than a mere topological space, for instance an $\mathcal{S}-$manifold, then we can view the assignment induced by its structure as the following object

$$\mathcal{S}_X(U) = \mathcal{S}(U)$$

the class of $\mathcal{S}-$functions defined from a certain open set U into $\mathbb{K}$. And this defines a subsheaf of the sheaf we encountered in the previous example, namely $\mathcal{C}_X$.

This sheaf appears in a wide variety of geometric situations, and due to its significance and importance (since it actually determines the class of functions one studies while doing geometry) it has a name, it is called the *structure sheaf*. For certain special classes of manifolds we have, for a certain manifold X the following

- In the smooth setting, we will use the *sheaf of differentiable functions* $\mathcal{C}_X^\infty = \mathcal{E}_X$.
- In the complex setting, we will usually use the *sheaf of holomorphic functions* $\mathcal{O}_X$.
- In the real-analytic case (that we will not study deeply) one is interested in the *sheaf of real-analytic functions* $\mathcal{A}_X$

Another important example that will be used later for the topological interest it has is the *constant sheaf*.

Example 2.4. Consider X to be a topological space and take G to be an abelian group. The *constant sheaf* is defined by the assignment of the abelian group G to every connected open subset U. All the inclusion morphisms $V \subset U$ get transformed by the sheaf into the identity

$$Id_G \colon G \to G.$$

The constant sheaf associated with an abelian group G will sometimes be denoted by $\underline{G}$ when we want to emphasize the distinction between the abelian group G and its associated constant sheaf. When there is no chance of confusion, the constant sheaf will also be denoted by G.

From all these examples one is tempted into a dangerous idea, to think that all the presheaves that one could find over an arbitrary topological space are actually sheaves, and thus there would be no need to make a distinction between the two notions. But unfortunately, **not every presheaf is a sheaf** and this forces us to make a distinction between both concepts. Next we will introduce an example of a presheaf that fails to be a sheaf to concretely see where a presheaf can fail to satisfy the sheaf axioms, and the possible obstructions to this.

Example 2.5. Consider our topological space X to be the complex plane $X = \mathbb{C}$ with the usual topology induced by the euclidean norm $\|\cdot\|_2$. We consider the *presheaf of bounded holomorphic functions* H^∞ over X defined by assigning to every open set $U \subset X$ the algebra of bounded holomorphic functions defined in the open set U, denoted by $H^\infty(U)$. The restriction maps of the presheaf, since we are dealing with actual functions, are just the usual restriction of functions, and as we saw before this defines a presheaf in a clear way. Now we want to see how the presheaf H^∞ **fails to be a sheaf**.

Consider the following open cover of the complex plane

$$U_i = \{z \colon |z| < i\}$$

for i being a positive integer. Clearly this open subsets cover the whole complex plane. Now we will see how this presheaf violates the second sheaf axiom. Define for every U_i the function

$$\begin{aligned} f^i \colon U_i &\to \mathbb{C} \\ f^i(z) &= z. \end{aligned}$$

Now clearly for every open set U_i of our cover, this $f^i(z)$ define bounded analytic functions on U_i, and thus $f^i \in H^\infty(U_i)$ for every positive integer i.

Now, if our presheaf H^∞ were actually a sheaf, we should be able to find a section $f \in H^\infty(X) = H^\infty(\mathbb{C})$ such that restricted to every U_i we have

$$f \restriction_{U_i} = f^i.$$

But we know from Liouville's theorem from one-dimensional complex analysis that the only bounded holomorphic functions on the whole complex plane (*entire functions*) are constants. Thus since $f \in H^\infty(X) = H^\infty(\mathbb{C})$ we have that by the theorem f is just a constant function, but then it cannot agree with the functions f^i when restricted since the restriction will also be constant but by definition the f^i's are not constant. Thus we have seen that $H^\infty(X)$ is a **presheaf that fails to be a sheaf**.

We can extract some knowledge from this failure, since precisely, the main reason implying that H^∞ is not a sheaf is that it is *not defined by a local property* such as differentiability, continuity, holomorphicity...

There are other examples of presheaves that fail the first sheaf axioms that we will not present here, since they are usually mainly "proof of concept" examples, while presheaves that fail the second sheaf axiom are usually more common in geometric situations.

Until now we have seen some examples where (pre)sheaves actually have some attached algebraic structure, such as Abelian Groups or $\mathbb{K}-$algebras. Now we turn to a definition of a usual structure that appears on many presheaves that are of geometric interest, the structure of *modules*. These structure can be viewed in many ways, as generalizations of Abelian groups ($\mathbb{Z}-$modules) or Vector spaces (modules over a field). We will work under the hypotheses that in our geometric situation we have a certain presheaf $\mathcal{R}$ of commutative rings. One can think for instance of $\mathcal{R}$ as being just the structure sheaf for a certain manifold, although more examples are possible. And a presheaf of abelian groups $\mathcal{M}$ that one can think of as being sections of a bundle for instance.

Definition 2.1.5. Let X be a topological space, $\mathcal{R}$ a presheaf of commutative rings over X and $\mathcal{M}$ a presheaf of abelian groups over X. Suppose that for any open set $U \subset X$ we can induce in $\mathcal{M}(U)$ the structure of an $\mathcal{R}(U)-$module such that for $\alpha \in \mathcal{R}(U)$ and $\sigma \in \mathcal{M}(U)$,

$$r_V^U(\alpha f) = \rho_V^U(\alpha) r_V^U(f)$$

for every pair of open sets $V \subset U$, where the maps $\{r_V^U\}$ are the restriction homomorphisms of $\mathcal{M}$ and the maps $\{\rho_V^U\}$ are the restriction homomorphisms of $\mathcal{R}$. If we are in this situation $\mathcal{M}$ is called a *presheaf of* $\mathcal{R}$*-modules*. Moreover, if $\mathcal{M}$ satisfies the properties of a sheaf, it is called a *sheaf of* $\mathcal{R}$*-modules*.

This definition, although it may seem abstract at first, encompases a wide variety of geometrical situations, and further generalizes the notion of section of a certain bundle, that was itself a generalization from the point of view of classical XIX'th century differential geometry. Hence we are step by step seeing how sheaf-theoretic notions allow us to put classical differential geometric notions into an abstract framework. We make the idea more concrete in the following example.

Example 2.6. Suppose that we have a certain $\mathcal{S}-$bundle (vector bundle) $\pi\colon E \to X$. We consider the following associated presheaf $\mathcal{S}(E) = \mathcal{S}_X(E)$ that is defined by

$$\begin{aligned} \mathcal{S}_X(E)\colon Open(X) &\to \mathcal{R} - mod \\ U &\to \mathcal{S}(U, E), \end{aligned}$$

defined by the differential-geometric sections of the bundle and with the natural restriction of sections. They turn out to be precisely $\mathcal{R}-$modules for $\mathcal{R}$ being the structure sheaf. For instance in the case of smooth manifolds, the sections of a certain bundle, that are usually denoted in that context by $\Gamma(E)$ form a $\mathcal{C}^\infty(X)-$module. Thus we can see how the sheaf-theoretic concepts encompass all this situations. The sheaf $\mathcal{S}_X(E)$ can be seen as a subsheaf of the sheaf of continuous functions $\mathcal{C}_X(E)$, the sheaf $\mathcal{S}_X(E)$ is usually called the *sheaf of $\mathcal{S}$-sections of the vector bundle E.*

Special cases of this general example include a variety of familiar and useful objects for many branches of geometry, namely:

- The sheaves of differential forms $\mathcal{E}_X^* = \Omega^*(X)$ on a differentiable manifold.
- The sheaves of differential forms of type $(p, q), \Omega^{p,q}(X) = \mathcal{E}_X^{p,q}$ on a complex manifold X.

From a complex-analytic perspective, where the study of sheaves came to importance for dealing with problems in complex geometry, one uses heavily certain types of sheafs, and those have a name.

Definition 2.1.6. For a complex manifold X, any sheaf of modules over the structure sheaf $\mathcal{O}_X$ of X is called an *analytic sheaf*.

Note that there is a natural and clear notion of *restriction* of a (pre)sheaf $\mathcal{F}$ on a certain topological space X to a (pre)sheaf on a certain open subset $U \subset X$. The definition just comes from the definition of the topology in U, thus using the (pre)sheaf $\mathcal{F}$ in the open sets of U. This restriction is called the *restriction sheaf*, and is denoted by $\mathcal{F}|_U$.

With this notion of restriction in our hands, we want to make a very important definition. One should note that this definition is not at all arbitrary and just follows from an algebraic hint. One knows from algebra that when studying modules, the simplest type of modules are those called free modules, and we want to make the corresponding definition for sheaves.

Definition 2.1.7. Let $\mathcal{R}$ be a sheaf of commutative rings over a certain topological space X.

- We define the presheaf $\mathcal{R}^p$, for $p \geq 0$ by the assignment

$$U \to \mathcal{R}^p(U) = \bigoplus_{j=1}^{p} \mathcal{R}(U).$$

  Given this definition, $\mathcal{R}^p$ is a sheaf of $\mathcal{R}-$modules. It is usually called the *direct sum* of the sheaf of commutative rings $\mathcal{R}$.

- If a certain sheaf of $\mathcal{R}-$modules $\mathcal{F}$ is isomorphic to the direct sum, meaning $\mathcal{F} \cong \mathcal{R}^p$ for some positive integer $p \geq 0$. Then $\mathcal{F}$ is called a *free sheaf* of modules.

- If $\mathcal{F}$ is a sheaf of $\mathcal{R}-$modules such that for every point $x \in X$, we can find a neighbourhood U_x containing x where the restricted sheaf $\mathcal{F} \restriction_{U_x}$ is free, then the sheaf $\mathcal{F}$ is called *locally free*.

Up until now we have seen sheaves as a kind of generalization of the notion of a bundle, but now we have the needed concepts to be able to make this correspondence more precise. With the following theorem one sees that using sheaves, one gains a more general language to deal with geometric problems, that includes as well the notion of a bundle, as was introduced in the previous chapter.

Theorem 2.1.1. *Consider the (connected) topological space X to be an $\mathcal{S}-$manifold, with $\mathcal{S}$ being the geometric structure. Then we have bijective correspondence between (isomorphism classes of) $\mathcal{S}-$(vector) bundles over X and (isomorphism classes of) locally free sheaves of $\mathcal{S}-$modules over X.*

Proof. Clearly the correspondence, as we said before, intuitively has to come from the assignment

$$E \to \mathcal{S}(E),$$

associating to each bundle its sheaf of sections. We have to see that this sheaf is in fact a locally free sheaf of $\mathcal{S}-$modules. To see this, we just have in our hands the local triviality of the bundle, and this is what we will use.

By the local triviality, around any point $x \in X$, we have an open neighborhood $x \in U$ such that $E|_U \cong U \times \mathbb{K}^r$, with r being the rank of the vector bundle E. From this it follows that the space of sections over the open set U has a clear isomorphism

$$\mathcal{S}(E)|U \cong \mathcal{S}(U \times \mathbb{K}^r),$$

and we claim that we also have

$$\mathcal{S}(U \times \mathbb{K}^r) \cong \mathcal{S}|_U \oplus \cdots \mathcal{S}|_U.$$

To see this, we start from the definition of a section, from which it follows that for $f \in \mathcal{S}(U \times \mathbb{K}^r)(V)$ (for $V \subset U$ open) if and only if $f(x) = (x, g(x))$, where $g\colon V \to \mathbb{K}^r$ and

g is an $\mathcal{S}-$morphism. Knowing this, it is clear that $g = (g_1, ..., g_r)$ for $g_j \in \mathcal{S}(V)$, and the isomorphism above is given by the correspondence

$$f \to (g_1, ..., g_r) \in \mathcal{S}|_U \oplus \cdots \mathcal{S}|_U,$$

which is the sheaf isomorphism that justifies the claim. And hence we have seen that the sheaf of sections of the bundle, $\mathcal{S}(E)$ is a locally free $\mathcal{S}_X-$module, as we wanted to see.

Now we have to go in the opposite direction, and construct a vector bundle from a certain locally free sheaf, thus inverting the above construction. To do this suppose that $\mathcal{L}$ is a locally free sheaf of $\mathcal{S}-$modules. The fact that the sheaf is locally free means that we can find an open covering $\{U_\alpha\}$ of X and an associated family of maps g_α such that

$$g_\alpha \colon \mathcal{L}|_{U_\alpha} \to \mathcal{S}^r|_{U_\alpha},$$

is an isomorphism for some $r > 0$. This just follows from the definition of a locally free sheaf. Note that since we are assuming that X is connected we have that r doesn't depend on α.

To construct the bundle associated to this sheaf we have to construct it's transition functions. We define

$$g_{\alpha\beta} \colon \mathcal{S}^r|_{U_\alpha \cap U_\beta} \to \mathcal{S}^r|_{U_\alpha \cap U_\beta},$$

by setting $g_{\alpha\beta} = g_\alpha \circ g_\beta^{-1}$, and since both g_α and g_β are isomorphisms, we deduce that the maps $g_{\alpha\beta}$ as defined are also isomorphisms.

Now, note that $g_{\alpha\beta}$ are sheaf mappings, and so in particular (when it acts on the open set $U_\alpha \cap U_\beta$) it determines an invertible mapping of vector-valued functions $(g_{\alpha\beta})_{U_\alpha \cap U_\beta}$ that we write as

$$g_{\alpha\beta} \colon \mathcal{S}(U_\alpha \cap U_\beta)^r \to \mathcal{S}(U_\alpha \cap U_\beta)^r,$$

this is then no more than a non-singular $r \times r$ matrix of functions in $\mathcal{S}(U_\alpha \cap U_\beta)$, meaning that

$$g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to GL(r, \mathbb{K}),$$

and this determines the transition functions that we are looking for. Clearly the compatibility conditions

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$$

are satisfied by the definition of g. With this in our hands we can thus define a vector bundle E by letting

$$\hat{E} = \dot{\bigcup}_\alpha U_\alpha \times \mathbb{K}^r,$$

the dot meaning that we are taking the disjoint union. Now we just build the bundle E by making the identification

$$(x, \sigma) \ (x, g_{\alpha\beta}(x)\sigma),$$

whenever $x \in U_\alpha \cap U_\beta$ (with the intersection being non-empty). This is the usual way to construct bundles given the transition functions. One can also see that, by the way that we have used isomorphisms along the proof, the isomorphism classes of both bundles and sheafs are preserved, thus concluding the proof. □

With this result in our hands we see in a precise way how sheaves also encompass the notion of bundles, and thus one can equivalently think of bundles from the point of view of the usual definition $\pi\colon E \to X$ or just by looking at its sheaf of sections. The important fact that this theorem tells us is that both contain the same amount of information up to isomorphism!

To conclude this chapter, we want to introduce a further generalization of the notion of a locally free sheaf that can be of use in some geometric situations. Note that we will mainly be interested during this work in locally free sheaves arising from vector bundles, but for the study of function theory in complex manifolds, or even if one is interested in spaces with singularities (that we will not study), this generalization is of great importance. To define this type of sheaves we need the notion of an exact sequence of sheaves, that we will define in following sections. The reader not familiar with the notion can skip this definition now and return to it later if necessary, or just think of this exact sequence as being exact "at every point" (we will later define what this means). It should be remarked that exactness of sheaves is not the natural definition one might think of, but we will explain that later.

Definition 2.1.8. Let $\mathcal{F}$ be an analytic sheaf on a complex manifold X. This sheaf is said to be *coherent* if for every point $x \in X$, we can find a neighborhood U of the point x such that there is an exact sequence of sheaves over U:

$$\mathcal{O}^p|_U \to \mathcal{O}^q|_U \to \mathcal{F}|_U \to 0,$$

for some p and q.

Note here that this is in fact a generalization of locally free sheaves since, if we take $p = 0$, then we recover the other definition.

One of the main reasons that coherent sheaves turn out to be very fundamental objects, specially in the field of Algebraic Geomtry, is the fact that they form, unlike vector bundles, an *abelian category*. This means that one can take kernels, images and cokernels and one will end up with another coherent sheaf. That is one of the properties that motivates the definition. Although we will not use them a lot in this work, the interested reader is referred to [48] for more information about this sheaves and their importance.

2.2 Sheaves as topological spaces

With the notions we have introduced in the previous section, we kind of deduce that sheaves on topological spaces are a way to carry certain local information that we may have in our space. Sheaves appear naturally in a wide variety of geometric constructions. Now we want to shift a bit the point of view we take on sheaves.

During this section we will view them as topological spaces of a particular type. This will enable us to have another model of a sheaf that will enable us in following sections to define the appropriate homological algebra constructions for sheaves. As we shall see, we will also be able to "generate" sheaves from presheaves in a certain way (we will introduce the construction in this section). Hence there seem to be good reasons that motivate the alternative view about sheaves. We start with a pair of definitions.

Definition 2.2.1. Let X be a topological space, suppose we have a topological space Y together with a continuous surjective mapping

$$\pi\colon Y \to X,$$

with π a local homeomorphism. Then Y is called an *étalé space* over the topological space X.

Remark. With this definition, as with the fibre-bundle construction in geometry, one thinks of the space Y as being "over" X and the map π as being a kind of projection. This motivates the next definition, as an analog of the notion of section in differential geometry.

Definition 2.2.2. Let U be an open set of a topological space X. And consider that we have an étalé space $\pi\colon Y \to X$. As usual, a *section* will just be a map from the base space into the étalé space Y such that composed with the projection gives the identity. We denote the set of sections as

$$\Gamma(U, Y) = \{\phi\colon U \to Y | \pi \circ \phi = Id_U\},$$

where the maps ϕ above are taken to be continuous with respect to the appropriate topologies in X and Y.

Remark. Clearly, the sections of a certain étalé space $\pi\colon Y \to X$ form a subsheaf of the sheaf of continuous functions $\mathcal{C}(X, Y)$.

It became clear in the previous section, with the introduction of the presehaf of bounded holomorphic functions $H^\infty(\mathbb{C})$ that, if a certain subsheaf is not defined by local data, then it may fail to be a sheaf. Hence the need to distinguish them. But suppose that our geometrical information is encoded in a certain subsheaf, are we doomed forever to remain at the "pre-level" or can we make a sheaf out of it? This is the question that we want to answer here. Briefly, to any presheaf $\mathcal{F}$ over a topological space X we will associate a certain étale space

$$\pi\colon \hat{\mathcal{F}} \to X.$$

The main requirement that we will need is that the sheaf of sections of $\hat{\mathcal{F}}$ gives just an equivalent model for $\mathcal{F}$ in the case that $\mathcal{F}$ is a sheaf. If not, we have assigned a certain sheaf (the sheaf of sections of the étalé space) to our original presheaf $\mathcal{F}$. As we develop this whole construction, the reasons motivating it will become clear, and we will rethink the whole construction at the end of the section.

To develop this construction, we will first need some essential concepts from sheaf theory, that allow us to further localize the information of certain (pre)sheaves to concrete points in our topological space. What one does is consider, for a given point in the topological space, equivalence classes of sections that agree near the point. This turns out to be a much fruitful concept that it may seem, since it generalizes notions such as local expansions (taylor-laurent series).

Definition 2.2.3. Let $\mathcal{F}$ be a presheaf over a topological space X. For every point $p \in X$ we consider the direct limit of $\mathcal{F}(U)$ with respect to the restriction maps $\{r_V^U\}$ of the sheaf $\mathcal{F}$. This construction is usually denoted by

$$\mathcal{F}_p \colon = \varinjlim_{x \in U} \mathcal{F}(U),$$

and is called the *stack* of $\mathcal{F}$ at the point p.

Remark. If the original presheaf has some algebraic structure, then the associated stacks will inherit this structure. Meaning that if for instance $\mathcal{F}$ is a (pre)sheaf of abelian groups then the stalks will also be abelian groups, and similarly for other structures.

There is a natural map

$$r_x^U : \mathcal{F}(U) \to \mathcal{F}_x,$$

for every point $x \in U$. This map takes some section in $\mathcal{F}(U)$ and sends it to its equivalence class in the direct limit. If we have a section $f \in \mathcal{F}(U)$, then $f_x \colon r_x^U(f)$ is called the *germ* of the section f at the point x, and the section f is called a *representative* for the germ f_x. The space that we will be interested in is

$$\hat{\mathcal{F}} = \bigcup_{x \in X} \mathcal{F}_x,$$

which is clearly equipped with a projection

$$\pi \colon \hat{\mathcal{F}} \to X$$

which just takes points in the stalk $\mathcal{F}_x$ to the associated point x. We want this space to really be an étale space, and thus we need to introduce a topology on it. To do this, for each section $f \in \mathcal{F}(U)$, we define the set function

$$\hat{f} \colon U \to \hat{\mathcal{F}}$$

by setting $\hat{f}(x) = f_x$ for every point $x \in U$. We note that we have $\pi \circ \hat{f} = Id_U$.

With this notions under our belt, we take, as a basis for the topology of $\hat{\mathcal{F}}$

$$\{\hat{f}(U)\},$$

for every open set $U \subset X$ and every section $f \in \mathcal{F}(U)$. Under this topology, just by construction, all the functions $\hat{f}$ are continuous functions.

Moreover, one can check that with this topology on $\hat{\mathcal{F}}$ the function π is continuous and indeed a local homeomorphism, thus making $\pi \colon \hat{\mathcal{F}} \to X$ into an étale space.

With the construction that we just made, we have associated to each presheaf $\mathcal{F}$ over the topological space X an étale space. And moreover, if the presheaf happens to have some algebraic properties which are preserved by direct limits, then the étale space that we constructed will inherit this properties.

In doing this association of the étale space $\hat{\mathcal{F}}$ to a presheaf $\mathcal{F}$, we have also associated, as we said before, a sheaf to our presheaf $\mathcal{F}$, precisely the sheaf of sections of the étale space $\hat{\mathcal{F}}$. This associated sheaf is usually called the *sheaf generated by* $\mathcal{F}$. But now the natural question arises, what is the relationship between the sheaf generated by $\mathcal{F}$ and the original (pre)sheaf $\mathcal{F}$? For the time being, and to explore the relationship between both, we will denote by $\overline{\mathcal{F}}$ the sheaf generated by $\mathcal{F}$.

To start exploring this relationship, one should start by noting that there is a presheaf morphism

$$\tau \colon \mathcal{F} \to \overline{\mathcal{F}},$$

which is given by

$$\tau_U \colon \mathcal{F}(U) \to \overline{\mathcal{F}}(U)[: \; = \Gamma(U, \hat{\mathcal{F}})],$$

with $\tau_U(f) = \hat{f}$. Recall that $\hat{f}(x) = r_x^U(f)$ for every $x \in U$.

Hence we have seen the relationship between the (pre)sheaf $\mathcal{F}$ and the sheaf that it generates, via this morphism of (pre)sheaves. But now the question arises, what if we had originally a sheaf to start with? What relationship does there exist in this case between both sheaves? And the answer is quite satisfactory as the following result tells us.

Theorem 2.2.1. *If $\mathcal{F}$ is a sheaf, then the morphism*

$$\tau \colon \mathcal{F} \to \overline{\mathcal{F}}$$

as defined above is a sheaf isomorphism.

The proof of this theorem can be seen in [1], since we don't reproduce it here, the reader which is interested in the proof is referred to that book. The proof has no special idea behind it, it is just an application of the sheaf axioms to the construction given above.

And thus with this construction, one can associate an étale space to any presheaf on our topological space. And by considering the sheaf of sections of this étale space, one gets a sheaf in a natural way. Furthermore, if the original presheaf was in fact a sheaf, then we have the same amount of information in both the étale space (and its sheaf of sections) and in the original sheaf. This gives us an important notion, the fact that sheaves can be thought of in two different ways which are equivalent following the previous theorem. This is what motivates the fact that in some books one sees sheaves defined as an étale space with certain algebraic structure along its fibres.

From the perspective of analysis, the main object will be the (pre)sheaf, viewed with its axioms as introduced in the beginning of the chapter. This is the more natural perspective since in analysis most sheaves appear naturally in this form. This doesn't mean that the étale space construction that was introduced in this section may be disregarded, since it is an auxiliary construction which is very useful for constructing the homological machinery that we are going to introduce in the following section.

2.3 Resolutions of sheaves

While dealing with topological and geometric questions in several areas one sees that *homological algebra* is a great tool. In order to carry geometric information and to better understand the geometry and topology of spaces the techniques of homological algebra are fundamental. To this end, we will now devote a bit of time to the theory of *homological algebra of sheaves* (of abelian groups). The concepts that we introduce can be generalized in the natural manner to sheaves of modules.

We begin by the definition of a standard construction in many areas of mathematics, the *quotient*.

Definition 2.3.1. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups over a certain topological space X,

$$\begin{aligned}\mathcal{F}&\colon Open(X) \to Ab\\ \mathcal{G}&\colon Open(X) \to Ab,\end{aligned}$$

with $\mathcal{G}$ being a subsheaf of $\mathcal{F}$. We consider Σ to be the sheaf generated by using the presheaf generated by the assignment

$$U \to \mathcal{F}(U)/\mathcal{G}(U).$$

This sheaf Σ is called the *quotient sheaf* of $\mathcal{F}$ by $\mathcal{G}$ and is denoted usually by $\mathcal{F}/\mathcal{G}$.

Note the important fact, which may be overlooked at first sight but which is of fundamental importance for what follows, that we are defining the quotient by using the étale space construction of the previous section. If one just naively considered the quotient of two sheaves, one may think that we should get a sheaf, but this is not always the case, thus the need for considering the associated generated sheaf. And this is only possible once one knows about the étale space construction.

The quotient assignment that we used in the definition above induces a natural surjection

$$\mathcal{F} \to \mathcal{F}/\mathcal{G},$$

by passing to the direct limit,thus inducing a continuous map between the associated étale spaces, and then considering the induced map on the continuous sections. This map is then the desired sheaf morphism onto the quotient sheaf.

In the field of homological algebra, one of the fundamental objects at play is the notion of *exactness* of sequences of maps. These are usually naturally defined in any abelian category but for sheaves the definition is a bit tricky, since we do not require *exactness* at every open set, as one could think would be the natural definition.

Definition 2.3.2. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three sheaves of abelian groups over a topological space X. Suppose that we have a sequence of sheaf morphisms

$$\mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}.$$

We say that this sequence of sheaf morphisms is *exact at* $\mathcal{B}$ is for every point $x \in X$ the induced sequence on the stalks

$$\mathcal{A}_x \xrightarrow{\alpha_x} \mathcal{B}_x \xrightarrow{\beta_x} \mathcal{C}_x$$

is exact, recall that in this case the stalks also have the structure of abelian groups inherited from that of the sheaves $\mathcal{A}, \mathcal{B}, \mathcal{C}$. As usual in homological algebra, a *short exact sequence of sheaves* is a sequence of sheaf morphisms

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

that is exact at $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, here 0 just denotes the (constant) zero sheaf.

Remark. It is very important to note that this notion of exactness **is a local property**. As we said before, we do not require the sheaves to be exact at the presheaf level, meaning that the sequence

$$\mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U),$$

may not be exact for every open set $U \subset X$. Which could have been a possible definition of exactness. But precisely it is important to know that sheaf-theory becomes a useful tool precisely because it is able to detect, find and categorize the obstructions one may have in a given geometric situation to the *global exactness* of sheaves.

One of the best ways to learn how this exact sequences behave is to look at particular examples.

Example 2.7. Suppose that X is a connected complex manifold. We denote by $\mathcal{O}$ the sheaf of holomorphic functions on X and by $\mathcal{O}^*$ the sheaf of nonvanishing holomorphic functions on X. $\mathcal{O}^*$ is a sheaf of abelian groups under multiplication. In this setting we have the following sequence:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{O} \xrightarrow{exp} \mathcal{O}^* \to 0,$$

where $\mathbb{Z}$ is the constant sheaf of integers and ι the natural inclusion. The other map is just the exponential map $exp\colon \mathcal{O} \to \mathcal{O}^*$ defined by

$$exp_u(f)(z) = exp(2\pi i f(z)).$$

For some (sufficiently small) simply-connected open neighborhood U of $x \in X$ and for some representative $g \in \mathcal{O}^*(U)$ of a certain germ g_x at the point x, we can choose the germ $f_x = ((\frac{1}{2\pi i})log(g))_x$ for a certain branch of the logarithm. With this we have that $exp_x(f_x) = g_x$. We also have that $exp_x(f_x)) = 0$ (0 being the identity element of the abelian group) implies that

$$exp(2\pi i f(z)) = 1, \;\; z \in U,$$

for any section $f \in \mathcal{O}(U)$ which is a representative of the germ f_x on a certain connected neighborhood U of x. Therefore we have that f is constant on U ans is in fact an integer, thus meaning that

$$Ker(exp_x) = \mathbb{Z},$$

and thus we have seen that the sequence is exact.

Example 2.8. Another natural example of an exact sequence is given by the quotient, in a similar fashion as it happens in many algebraic categories. If $\mathcal{A}$ is a subsheaf of a sheaf $\mathcal{B}$, then

$$0 \to \mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{\pi} \mathcal{B}/\mathcal{A} \to 0,$$

is an exact sequence of sheaves where ι is the natural inclusion and π the natural quotient mapping.

Now we wish to incorporate more of the useful notions of homological algebra when dealing with sequences of morphisms. In this context we will mainly refer to a sheaf either as a sheaf of abelian groups or a sheaf of modules, meaning that we impose that the sheaves that we work with have some algebraic structures to use while doing homological algebra with sheaves. Mainly because of the fact that we want to be able to take quotients, kernels, cokernels...

Definition 2.3.3. We define a *graded sheaf* as a family of sheaves $\mathcal{F}^* = \{\mathcal{F}^j\}_{j\in\mathbb{Z}}$ indexed by integers.

Definition 2.3.4. Let $\mathcal{F}^* = \{\mathcal{F}^j\}_{j\in\mathbb{Z}}$ be a graded sheaf. If this graded sheaf is furthermore connected by a sequence of sheaf mappings $\alpha_j\colon \mathcal{F}^j \to \mathcal{F}^{j+1}$

$$\cdots \to \mathcal{F}^0 \xrightarrow{\alpha_0} \mathcal{F}^1 \xrightarrow{\alpha_1} \mathcal{F}^2 \xrightarrow{\alpha_2} \mathcal{F}^3 \to \cdots,$$

then we call the graded sheaf $\mathcal{F}^*$ together with the family of maps α_j a *sequence of sheaves* (or sheaf sequence).

Now that we have one of the main objects of homological algebra that appears in geometric settings under our belt for sheaves, we can consider a special case of sequences of sheaves that generalize to a more abstract settings differential relations such as the property $d^2 = 0$ of the exterior derivative acting on differential forms.

Definition 2.3.5. A sequence of sheaves $(\mathcal{F}^*, \alpha_*)$ such that any composition

$$\mathcal{F}^{j-1} \xrightarrow{\alpha_{j-1}} \mathcal{F}^j \xrightarrow{\alpha_j} \mathcal{F}^{j+1}$$

is just the zero homomorphism ($\alpha_j \circ \alpha_{j-1} = 0$) is called a *differential sheaf.*

And there is one more notion that we will need from homological algebra, the notion of a *resolution*. Resolutions are very useful in a lot of areas of geometry where one can apply techniques from homological algebra.

Definition 2.3.6. Suppose we have a sheaf $\mathcal{F}$ over a topological space X, if we can find an exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \to \mathcal{F}^n \to \cdots,$$

we call this a *resolution* of the sheaf $\mathcal{F}$ and we abbreviate the sequence by using the notation

$$0 \to \mathcal{F} \to \mathcal{F}^*.$$

Resolutions turn out to be a very useful tool (as in other places of geometry) for the study of sheaves since, as we will see in later sections, the knowledge of a given resolution for a certain sheaf $\mathcal{F}$ can allow us to deduce information of various types about the sheaf $\mathcal{F}$. We will also later see how they can come useful when one deals with the *cohomology of sheaves.*

Now we want to give some examples of resolutions of some sheaves that are very important when studying differential geometry and also one example that connects with algebraic topology. The purpose of this examples is to see that although in the abstract sheaf-theoretic setting resolutions can seem abstract objects having nothing to do with geometry, they are actually just the right theoretical framework to be able to put a common language to a variety of different geometric constructions. It just took some years for mathematicians to realize that the sheaf-theory language was precisely the appropriate to construct a common language, an idea widely spread by some of the best french mathematicians of the past century (such as Leray, Cartan, Serre, Groethendick...).

Example 2.9. Resolution of the sheaf of differential forms

Let X be a (connected) differentiable manifold of real dimension m and consider the sheaf of real-valued differential forms of degree p, namely $\mathcal{E}^p_X$. Then we can find a resolution of the constant sheaf $\mathbb{R}$ given by

$$0 \to \mathbb{R} \xrightarrow{\iota} \mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^m_X \to 0,$$

where ι is just the natural inclusion and d is the exterior differential. Due to the fact that $d^2 = 0$ we have that the above is a differential sheaf, in fact this is one of the fundamental examples motivating the abstract definition.

The exactness is just a consequence of the classical Poincaré lemma and the fact that a function f with $df = 0$ everywhere must be constant, since we are assuming the manifold is connected. We denote this resolution by

$$0 \to \mathbb{R} \to \mathcal{E}_X^*$$

or if we are using complex coefficients we denote it by

$$0 \to \mathbb{C} \to \mathcal{E}_X^*,$$

understanding that one is considering complex-valued differential forms.

Example 2.10. Resolution of the constant sheaf for an abelian group

Let X be a topological manifold and G an abelian group. We want to see how to derive a resolution for the constant sheaf G over X. We denote by $S^p(U,G)$ the group of singular cochains in U with coefficients in G, meaning

$$S^p(U,G) = Hom_{\mathbb{Z}}(S_p(U,\mathbb{Z}), G),$$

where $S_p(U,\mathbb{Z})$ is the usual abelian group of integral singular chains of degree p in U with the usual boundary map that is introduced in algebraic topology. If we denote by $\delta\colon S^p(U,\mathbb{Z}) \to S^{p+1}(U,\mathbb{Z})$ the usual coboundary operator, then we denote by $S^p(G)$ the sheaf over X generated by the presheaf

$$U \to S^p(U,G),$$

endowed with the differential mapping $S^p(G) \xrightarrow{\delta} S^{p+1}(G)$.

If we consider U to be the unit ball in Euclidean space, then the sequence

$$\cdots \to S^{p-1}(U,G) \xrightarrow{\delta} S^p(U,G) \xrightarrow{\delta} S^{p+1}(U,G) \to \cdots,$$

is exact since $Ker(\delta)/Im(\delta)$ is just the classical singular cohomology for the open unit ball, and since this is contractible, it is a result from algebraic topology that this is zero for $p > 0$. From this we can conclude that the sequence

$$0 \to G \to S^0(G) \xrightarrow{\delta} S^1(G) \xrightarrow{\delta} S^2(G) \to \cdots \to S^m(G) \to \cdots,$$

is a resolution of the constant sheaf G, by noting that

$$Ker(\delta\colon S^0(U,G) \to S^1(U,G)) \cong G.$$

One should note here that if X is a differentiable manifold, then we can also condier $\mathcal{C}^\infty$ chains, meaning (linear combinations of) maps $f\colon \Delta^p \to U$, where f is required to be a smooth mapping defined in a neighborhood of the standard $p-$simples Δ^p.

The corresponding results above also hold in this case, and we thus have a resolution of the constant sheaf G by differentiable cochains with coefficients in G:

$$0 \to G \to S^0_\infty(G) \xrightarrow{\delta} S^1_\infty(G) \xrightarrow{\delta} S^2_\infty(G) \to \cdots \to S^m_\infty(G) \to \cdots ,$$

which we can abbreviate by

$$0 \to G \to S^*_\infty(G).$$

Although a resolution by smooth chains is more adapted to our geometric setting, one should note that the theory of singular chains has the ability to tackle problems for arbitrary topological spaces, while smooth chains are only defined provided that one has some additional structure. Since we will always be under this situation in this work, smooth chains will be useful for us, since they are easier to relate to other geometric objects. But if one is interested in more general topological spaces, one has to use the more general algebro-topolocial theory. It is a result in algebraic topology that at the level of cohomology both agree, and thus we will not talk much more about general non-smooth chains during this work.

Before we continue to the next examples, we will need a technical lemma that is kind of the analog of the classical Poincaré lemma for real differential forms in the complex setting. This lemma is usually called the $\overline{\partial}-$Poincaré lemma or alternatively the Dolbeaut-Grothendieck lemma.

Lemma 2.3.1. *Let ω be a $(p,q)-$form defined in a polydisk $\Delta \subset \mathbb{C}^n$, where $\Delta = \{z \in \mathbb{C}^n \colon |z_i| < r, i = 1, ..., n\}$. If ω is $\overline{\partial}-$closed in Δ, meaning that $\overline{\partial}\omega = 0$, then there exists a $(p, q-1)-$form η defined in a (slightly) smaller polydisk $\Delta' \subset\subset \Delta$, so that*

$$\overline{\partial}\eta = \omega$$

for every point in Δ'.

We omit the proof, the interested reader can check (Chap II. Lemma 2.15) in [1].

From this one deduces that the same holds true for the operator ∂, as one can check by using conjugation.

Now that we have already seen some examples of sequences of sheaves, differential sheaves, and resolutions, we are ready to introduce the natural notion of morphisms between differential sheaves.

Definition 2.3.7. Let $\mathcal{A}^*$ and $\mathcal{B}^*$ be two differential sheaves. We say that

$$f\colon \mathcal{A}^* \to \mathcal{B}^*$$

is a *homomorphism* (of differential sheaves) if we have a family of morphisms

$$f_j \colon \mathcal{A}^j \to \mathcal{B}^j$$

commuting with the differentials of the respective sequences of maps in $\mathcal{A}^*$ and $\mathcal{B}^*$, as usual in algebraic topology when defining operations with (co)-chain complexes, one has to impose commutativity with the appropriate differential maps.

Definition 2.3.8. In a similar fashion as in the definition above, we define a *homomorphism of resolutions* of sheaves $\mathcal{A}$ and $\mathcal{B}$ as a homomorphism of the underlying differential sheaves that makes the following diagram commute

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A}^* \arrow[d] \\ 0 \arrow[r] & \mathcal{B} \arrow[r] & \mathcal{B}^* \end{tikzcd}$$

Now we are going to see how this abstract conept generalizes to a well-known example in differential geometry, where when dealing with *de-Rahm theory*, one is interested in relating differential forms on a manifold to topological information about it. The ideas that we have introduced, together with some others still to come (mainly the big machinery of *sheaf cohomology*), allow us to understand some classical theorems in this language, and give alternative proofs of the theorems in this new language. For instance, for the well-known *de-Rahm isomorphism*, it was the french mathematician Henri Cartan who, in correspondence with André Weil, developed a new proof of this classical theorem. Once we have in our hands all the technichal machinery we will show the idea of how this reformulation of the proof goes. The important thing to know here is that although the theorem is a very concrete result in differential geometry, the techniques that can be used to prove it fall under this general sheaf-theoretic mindset, and thus can be applied to a wide variety of geometric situations, while the *core ideas* remain the same. Thus one can see that using almost the same proof one can get also similar results for complex differential forms, a result that was originally due to Pierre Dolbeaut, a french mathematician who studied under the supervision of Cartan.

Up to this point we can relate various sheaf resolutions over our manifold. We still need the theory of *sheaf cohomology* to be able to make sense out of this relationship, but we are closer to our goal.

2.4 Cohomology theory for sheaves

In this section we will give a quick review of *sheaf cohomology*. We will mainly quote the fundamental results of the theory that will be of use later and refer the reader to [1] for the proofs. Our main interest will be in applying the theory to some concrete problems.

We said before that cohomology of sheaves tries to capture the possible difficulties in relating local information to global information, we want to see how does the theory do this.

Suppose that we have a short exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

then one can see by unraveling the definition that the induced sequence

$$0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X) \to 0,$$

is exact both at $\mathcal{A}(X)$ and $\mathcal{B}(X)$ but not necessarily at $\mathcal{C}(X)$. And cohomology tries to capture in a precise sense the how this sequence fails to be exact at $\mathcal{C}(X)$.

First, we will start by introducing a class of sheaves where this problem can be solved, and later we will define cohomology in terms of this special sheaves.

If $\mathcal{F}$ is a sheaf over a topological space X, and we have a closed subset $S \subset X$, then we define

$$\mathcal{F}(S) \colon = \varinjlim_{S \subset U} \mathcal{F}(U),$$

where this direct limit runs as usual over the open sets U containing S and is taken with respect to the inclusions. If one looks at it from the perspective of the étale space introduced in the previous section, then $\mathcal{F}(S)$ can be identified with (continuous) sections of $\overline{\mathcal{F}}|_S$.

From now on, we will suppose that we are dealing with sheaves of abelian groups over a paracompact Hausdorff space. This is mainly a technical requirement, and note that for us is not much of a restriction, since manifolds satisfy this requirements.

Definition 2.4.1. A sheaf $\mathcal{F}$ over X such that for any closed subset $S \subset X$ the restriction

$$\mathcal{F}(X) \to \mathcal{F}(S),$$

is surjective, is called a *soft* sheaf.

Remark. Note that the definition just means that any section of the sheaf over S can be extended to a global section on the whole space X.

One of the fundamental reasons that makes soft sheaves important is that there are no obstructions to the exactness problem we saw before.

Theorem 2.4.1. *If $\mathcal{A}$ is a soft sheaf and we have a short exact sequence of sheaves*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

then the induced sequence

$$0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X) \to 0,$$

is also exact.

For the interested reader, this is (Chap II. Theorem 3.2) in [1].

Before analyzing the consequences of this theorem, we will introduce another class of sheaves, that are also very important from the geometrical point of view and that will give us a lot of examples of soft sheaves. Note that by using the abstract definition it is not trivial no know if a given sheaf is soft or not.

Definition 2.4.2. A sheaf $\mathcal{F}$ over X is called *fine* if for any locally finite open cover $\{U_i\}$ of the base space X we can find a family of sheaf morphisms

$$\{\tau_i \colon \mathcal{F} \to \mathcal{F}\},$$

satisfying

1. $\sum \tau_i = 1$.
2. $\tau_i(\mathcal{F}_x)$ for every x in some neighborhood of the complement of U_i.

The family $\{\tau_i\}$ is called a *partition of unity.*

Note that this notion of partition of unity is just the same one uses in (real) differential geometry, but in the sheaf language. The usual topological and smooth partitions of unity induce, in a natural way, the required sheaf-theoretic partitions of unity. And thus one has a big family of examples of fine sheaves coming from the field of geometry.

Now, how do fine sheaves relate to soft sheaves? The following result tells us their relationship.

Lemma 2.4.1. *If $\mathcal{F}$ is a fine sheaf over X, then it is soft.*

For the interested reader, this is (Chap II. Proposition 3.5) in [1].

Here we note that one of the examples we had in the previous sections of sheaves, namely the constant sheaves are neither fine nor soft. This is due to the fact that if G is a constant sheaf over X, then one can take two distinct points $x, y \in X$, and define a section $f \in G(\{x\} \cup \{y\})$ by $f(a) = 0, f(b) \neq 0$. This section can not be extended to a global section of the constant sheaf, and thus G is not soft. By the previous lemma we thus deduce that the sheaf is not fine either.

Now we go back to the theorem above and extract some of its consequences.

Corollary. If $\mathcal{A}$ and $\mathcal{B}$ are both soft and we have

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

exact, then $\mathcal{C}$ is also soft.

Corollary. If
$$0 \to \mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \cdots$$
is an exact sequence, where all the sheaves are soft, then the induced sequence of global sections
$$0 \to \mathcal{A}_0(X) \to \mathcal{A}_1(X) \to \mathcal{A}_2(X) \to \cdots ,$$
is also exact.

With this facts under our belt, we can now construct the so-called *canonical soft resolution* of a sheaf. Let $\mathcal{A}$ be a sheaf and $\pi\colon \overline{\mathcal{A}} \to X$ be the associated étale space. Then we define a presheaf
$$\mathcal{C}^0(\mathcal{A})(U) = \{s\colon U \to \overline{\mathcal{A}}; \pi \circ s = Id_U\}.$$
This presheaf is in fact a sheaf and is called the *sheaf of discontinuous sections of* $\mathcal{A}$ *over* X. Note that this are not necessarily sections since, as we defined them, sections were required to be continuous, thus this is a bigger space. One has a natural injection
$$0 \to \mathcal{A} \to \mathcal{C}^0(\mathcal{A}).$$

With this objects, we can now define $\mathcal{F}^1(\mathcal{A}) = \mathcal{C}^0(\mathcal{A})/\mathcal{A}$, and $\mathcal{C}^1(\mathcal{A}) = \mathcal{C}^0(\mathcal{F}^1(\mathcal{A}))$. We can iterate this proces and define
$$\mathcal{F}^j(\mathcal{A}) = \mathcal{C}^{j-1}(\mathcal{A})/\mathcal{F}^j(\mathcal{A}),$$
and also
$$\mathcal{C}^j(\mathcal{A}) = \mathcal{C}^0(\mathcal{F}^j(\mathcal{A})).$$

With this construction, we have by definition two short exact sequences of sheaves:
$$0 \to \mathcal{A} \to \mathcal{C}^0(\mathcal{A}) \to \mathcal{F}^1(\mathcal{A}) \to 0,$$
and
$$0 \to \mathcal{F}^j(\mathcal{A}) \to \mathcal{C}^j(\mathcal{A}) \to \mathcal{F}^{j+1}(\mathcal{A}) \to 0.$$

From this two short exact sequences we can form the long exact sequence
$$0 \to \mathcal{A} \to \mathcal{C}^0(\mathcal{A}) \to \mathcal{C}^1(\mathcal{A}) \to \mathcal{C}^2(\mathcal{A}) \to \cdots ,$$
and this sequence is called the *canonical soft resolution* of the sheaf $\mathcal{A}$. The reason to call it soft is that the sheaf of discontinuous sections $\mathcal{C}^0(\mathcal{A})$ is soft.

Now that we know how to build this resolution, we are closer to defining the cohomology groups of a space with coefficients in a sheaf. In the same setting as above, with $\mathcal{A}$ being a sheaf over X, and considering the canonical resolution, by taking global sections we obtain
$$0 \to \Gamma(X, \mathcal{A}) \to \Gamma(\mathcal{C}^0(\mathcal{A})) \to \Gamma(\mathcal{C}^1(\mathcal{A})) \to \cdots \to \Gamma(\mathcal{C}^q(\mathcal{A})) \to \cdots ,$$

and this sequence forms a cochain complex (meaning the composition of two maps is zero), but not necessarily exact. This sequence is in fact exact at $\Gamma(X, \mathcal{A})$ by definition, and if furthermore the original sheaf $\mathcal{A}$ is soft, it is exact everywhere by the previous corollaries. We define

$$C^*(X, \mathcal{A})\colon = \Gamma(X, \mathcal{C}^*(\mathcal{A})),$$

thus allowing us to rewrite the cochain complex above as

$$0 \to \Gamma(X, \mathcal{A}) \to C^*(X, \mathcal{A}).$$

After all this construction, we are finally able to define the cohomology groups, which will just be the cohomology of the complex above.

Definition 2.4.3. Let $\mathcal{A}$ be a sheaf over X and define

$$H^q(X, \mathcal{A})\colon = H^q(C^*(X, \mathcal{A})),$$

with $H^q(C^*(X, \mathcal{A}))$ being just the derived subgroups of the cochain complex, meaning

$$H^q(C^*(X, \mathcal{A})) = \frac{Ker(C^q \to C^{q+1})}{Im(C^{q-1} \to C^q)},$$

where $C^{-1} = 0$. These abelian groups $H^q(X, \mathcal{A})$, which are defined for every $q \geq 0$ are called the *sheaf cohomology groups of X of degree q with coefficients in $\mathcal{A}$.*

There are various ways of representing and understanding this cohomology groups, and whole books have been devoted to the study of this groups. The point of view of Groethendick, for instance, is that one should keep the space X fixed, and look at cohomology as a functor. To do this one thinks of cohomology as a functor from sheaves of abelian groups to abelian groups, first by considering the functor that assigns to every sheaf (of abelian groups) its global sections:

$$\Phi\colon \mathcal{F} \to \mathcal{F}(X),$$

this functor is in general left exact but not right exact, and thus to account for this non-exactness, one considers the right derived functors associated with the functor Φ, and those are precisely the cohomology groups in Grothendieck's view. This abstract view is in fact the same as the more "down to earth" construction that we gave, and shows in a clear way that $H^0(X, \mathcal{A}) = \Gamma(X, \mathcal{A})$, and also all the functoriality properties that these cohomology groups satisfy.

The properties satisfied by this cohomology groups can be seen in [1] and we will not devote to their study, and only comment on some of the properties that we will in this work. We will state them now, the proofs can be seen in the book.

1. As we said before, $H^0(X, \mathcal{A}) = \Gamma(X, \mathcal{A}) = \mathcal{A}(X)$, the global sections.

2. If $\mathcal{A}$ is a soft sheaf, then $H^q(X, \mathcal{A}) = 0$ for $q > 0$.

3. Any sheaf morphism
$$\tau \colon \mathcal{A} \to \mathcal{B}$$
induces a group homomorphism for each $q \geq 0$
$$\tau_q \colon H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B}),$$
satisfying $\tau_0 = \tau_X \colon \mathcal{A}(X) \to \mathcal{B}(X)$, $\tau_q = Id$ if τ is the identity map and $\tau_q \circ \sigma_q = (\tau \circ \sigma)_q$, if $\sigma \colon \mathcal{B} \to \mathcal{D}$ is another sheaf morphism.

Resolutions that have trivial cohomology have a special name due to its importance.

Definition 2.4.4. A resolution C^* of a sheaf $\mathcal{A}$ over X
$$0 \to \mathcal{A} \to \mathcal{C}\surd$$
is called *acyclic* if $H^q(X, C^p) = 0$ for every $q > 0$ and $p \geq 0$.

A fine or soft resolution of a certain sheaf is automatically acyclic, due to the previous properties. By using acyclic resolutions, one can compute the cohomology groups of a sheaf, due to a result that is usually known as the *abstract de Rham theorem.*

Theorem 2.4.2. *If $\mathcal{A}$ is a sheaf over X and*
$$0 \to \mathcal{A} \to C^*$$
is a resolution of $\mathcal{A}$. Then we have natural homomorphisms for every p
$$\gamma^p \colon H^p(\Gamma(X, C^*)) \to H^p(X, \mathcal{A}),$$
with $H^p(\Gamma(X, C^*))$ being the derived group of the cochain complex $\Gamma(X, C^*)$. Moreover, if the resolution
$$0 \to \mathcal{A} \to C^*$$
is acyclic, then the homomorphisms γ^p are isomorphisms.

The interested reader can check the proof in (Chap. II Theorem 3.13) [1].

From this result we extract the following corollary.

Corollary. If
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d] & \mathcal{A}^* \arrow[d] \\
0 \arrow[r] & \mathcal{J} \arrow[r] & \mathcal{B}^*
\end{tikzcd}$$

is a homomorphism of sheaf resolutions. Then we have an induced homomorphism

$$H^p(\Gamma(X,\mathcal{A}^*)) \to H^p(\Gamma(X,\mathcal{B}^*)),$$

which moreover is an isomorphism provided that both resolutions are acyclic and that the map

$$\mathcal{S} \to \mathcal{J},$$

is an isomorphism

We can now see a neat application of all this theory. Just by using this corollary one can prove the usual de Rham theorem, relating singular (smooth) cohomology and the cohomology of differential forms, in the case of differentiable manifolds. This is an example of a result that was proven without sheaves, but that by using them, becomes much more structured.

Theorem 2.4.3. ***(de Rham)*** *Let X be a differentiable manifold. Then the natural mapping*

$$I\colon H^p(\mathcal{E}^*(X)) \to H^p(S^*_\infty(X,\mathbb{R}))$$

induced by integration of differential forms over the smooth singular chains with real coefficients, is an isomorphism.

Proof. We have two distinct resolutions of the constant sheaf $\mathbb{R}$, namely

$$0 \to \mathbb{R} \to \mathcal{E}^*$$

given by differential forms, and

$$0 \to \mathbb{R} \to S^*_\infty(\mathbb{R}),$$

given by singular smooth cochains.

By the existence of partitions of unity, we have that $\mathcal{E}^*$ is fine (and thus also soft). We also have that $S^*_\infty(\mathbb{R})$ is also soft. We want to see that $S^p_\infty(\mathbb{R})$ are also soft for every p. To see this, note that $S^p_\infty(\mathbb{R})$ is an $S^0_\infty(\mathbb{R})-$module.

We claim that $S^0_\infty(\mathbb{R})$ is soft. But this just easily follows from the fact that $S^0_\infty(\mathbb{R}) = S^0(\mathbb{R}) = \mathcal{C}^0(X,\mathbb{R})$, meaning that for each point in the manifold X (a singular $0-$cochain) we assign a real value. This shows us that $S^0_\infty(\mathbb{R})$ is soft. Now, from the fact that a sheaf of modules over a soft sheaf of rings is also soft (Chap II. Lemma 3.16 in [1]), we conclude that S^p is also soft.

Knowing that both resolutions are thus acyclic, and using the previous corollary, we conclude that the mapping I (induced by integration) is indeed an isomorphism! □

But the real power of sheaf theory comes now, since by using the same ideas (the *de Rham abstract theorem*) one deduces, by doing actually almost the same proof in the complex setting, the analogue of the de Rham theorem in the complex setting, which was proved by Dolbeaut.

Theorem 2.4.4. *If X is a complex manifold, then we have the isomorphism*

$$H^q(X, \Omega^p) \cong \frac{Ker(\mathcal{E}^{p,q}(X) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,q+1}(X))}{Im(\mathcal{E}^{p,q-1}(X) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,q}(X))}.$$

Proof. This just comes from the fact that the resolution of the sheaf of holomorphic functions is fine, and thus acyclic. Hence by applying the abstract de Rham theorem, we get the desired result. □

And this is the power of sheaf theory, getting the ability to see the common logic behind many results that at first sight can seem not so similar. In fact now we will generalize this Dolbeaut theorem to an arbitrary holomorphic vector bundle. But before we do that, we need to introduce the tensor product of sheaves of modules.

Definition 2.4.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two sheaves of modules over a certain sheaf of commutative rings $\mathcal{R}$. Then we denote by $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}$ the sheaf generated by the presheaf

$$U \to \mathcal{A}(U) \otimes_{\mathcal{R}(U)} \mathcal{B},$$

is called the *tensor product of $\mathcal{A}$ and $\mathcal{B}$.*

From this definition it follows that, at the level of stalks we have

$$(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})_x = \mathcal{A}_x \otimes_{\mathcal{R}_x} \mathcal{B}_x.$$

This equality at the stalk level implies the following.

Lemma 2.4.2. *If $\mathcal{B}$ is a locally free sheaf of $\mathcal{R}-$modules and we have a short exact sequence of $\mathcal{R}-modules$*

$$0 \to A' \to A \to A'' \to 0,$$

then by taking the tensor product we obtain that

$$0 \to A' \otimes_{\mathcal{R}} \mathcal{B} \to A \otimes_{\mathcal{R}} \mathcal{B} \to A'' \otimes_{\mathcal{R}} \mathcal{B} \to 0,$$

is also exact.

Recall that over a complex manifold X we have a resolution

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1} \xrightarrow{\overline{\partial}} \cdots \to \mathcal{E}^{p,n} \to 0.$$

And we know that if E is a holomorphic vector bundle, then $\mathcal{O}(E)$ is a locally free sheaf. By using the previous lemma we get the resolution

$$0 \to \Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E) \to \mathcal{E}^{p,0} \otimes_{\mathcal{O}} \mathcal{O}(E) \xrightarrow{\overline{\partial}\otimes 1} \mathcal{E}^{p,1} \otimes_{\mathcal{O}} \mathcal{O}(E) \xrightarrow{\overline{\partial}\otimes 1} \cdots \xrightarrow{\overline{\partial}\otimes 1} \mathcal{E}^{p,n} \otimes_{\mathcal{O}} \mathcal{O}(E) \to 0.$$

Note that we also have a naturally defined isomorphism

$$\Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E) \cong \mathcal{O}(\wedge^p T^*X \otimes_{\mathbb{C}} E),$$

and also

$$\Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E) \cong \mathcal{E}^{p,q} \otimes_{\mathcal{E}} \mathcal{E}(E) \cong \mathcal{E}(\wedge^{p,q} T^*X \otimes_{\mathbb{C}} E),$$

where $\mathcal{E}(E)$ just denotes the sheaf of differentiable sections of the (differentible) bundle E. This is a consequence of the fact that

$$\mathcal{O}(E) \otimes_{\mathcal{O}} \mathcal{E} = \mathcal{E}(E),$$

due to the fact that $\mathcal{E}^{p,q}$ is also a $\mathcal{E}-$module.

The elements of $\mathcal{O}(X, \wedge^p T^*X \otimes_{\mathbb{C}} E)$ are called (global) *holomorphic $p-$forms on X with coefficients in E*. For notational simplicity, it is usual to denote this by $\Omega^p(X, E)$, and we denote the sheaf by $\Omega^p(E)$.

In an analogous manner, we denote by

$$\mathcal{E}^{p,q}(X, E) \colon = \mathcal{E}(X, \wedge^{p,q} T^*X \otimes_{\mathbb{C}} E),$$

and those are called the differentiable $(p, q)-$forms on X with coefficients in E.

Thus, we can rewrite the resolution of the holomorphic forms with coefficients in E (letting $\overline{\partial}_E = \overline{\partial} \otimes 1$)

$$0 \to \Omega^p(E) \to \mathcal{E}^{p,0}(E) \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,1}(E) \xrightarrow{\overline{\partial}_E} \cdots \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,n}(E) \to 0.$$

And due to the fact that this resolution is a fine resolution, by applying the same argument as in the proof of the de Rham and Dolbeaut theorems, we get the following far reaching generalization of Dolbeaut's theorem.

Theorem 2.4.5. *If X is a complex manifold and $E \to X$ is a holomorphic vector bundle. Then we have an isomorphism*

$$H^q(X, \Omega^p(E)) \cong \frac{Ker(\mathcal{E}^{p,q}(X, E) \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,q+1}(X, E))}{Im(\mathcal{E}^{p,q-1}(X, E) \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,q}(X, E))}.$$

This result concludes this chapter, to sum it up, it is important to know that sheaf-theoretic notions have allowed us to find the common logic behind many theorems that a priori could seem as very unrelated. We have seen that the fundamental reason that they are true is, in a sense, the same, the fact that one can find "nice enough" resolutions for the objects one is interested in.

Before we conclude, one should note that sheaves, although being objects that have been in the domain of pure mathematics for a long time, can also be of use in many applied situations. Due to the fact that, in essence, they capture information in a topological space, and the local to global behaviour, they can be very useful in a lot of "real-life" problems. This has not been very common for many years, but now there is some work in this direction that we think deserves some mention. One of the mathematicians that has succesfully applied sheaf-theoretic ideas to many applied areas, such as Data analytics, Wireless Network analysis, graph theory and some others is Michael Robinson. We refer the interested reader to some of his work in those applied areas, such as [14]. He has even programmed a python module for using sheaf-theoretic techniques! There are also more works in this direction such as [51] and [52].

3 Complex differential geometry

In this chapter we will develop some of the ideas of Hermitian differential geometry, that are of fundamental importance both for geometers and physicists. Since Quantum Phyisics is fundamentally related with Unitary operators, and thus we need some Hermitian structure to make sense of this. Afterwards we will study connections on these Hermitian structures. We will also introduce some invariants called Chern Classes, that will also allow us to study some facts about special classes of bundles. Once this machinery is introduced, we will mainly focus on the case of compact manifolds, and specifically in so-called (compact) *Kähler manifolds*. We will do some representation-theoretic work for the algebra $\mathfrak{sl}(2,\mathbb{C})$ that we will later apply to get some results about the topology of these manifolds due to Lefschetz.

There will be two main applications of what we do at the end, one to the problem of characterizing manifolds that can be embedded into certain spaces, the so-called theory of *Stein manifolds* and the *Kodaira embedding theorem*, and later we will study a special class of metrics called *Kähler-Einstein*, and explain the role the Calabi Conjecture plays in their study. We will sketch the proof of the conjecture by Yau, leading to the notion of so-called *Calabi-Yau manifolds*, and comment a bit on their importance in physics.

3.1 Hermitian structures in differential geometry

In this section we want to introduce some of the differential-geometric tools that are available in the context of *holomorphic vector bundles*. More generally, what we will want to study are $\mathbb{C}-$vector bundles in the differentiable setting. One can find a resemblance to the theory of real bundles, but complex bundles will turn out to have some structure that we will need to uncover. We will introduce the notions of connections, curvatures and metrics in this setting.

While dealing with bundles in this setting by *vector bundle* we will understand that we have a base manifold X and a certain differentiable $\mathbb{C}-$ vector bundle over the base space X. That we usually denote by

$$\pi\colon E \to X.$$

One can construct similar theories to the ones we are going to construct in other geometric settings, such as $\mathbb{R}-$vector bundles, and we will briefly latter comment on what kind of structures does one find in other geometric settings when one wants to study the geometric and topological properties of certain manifolds. Since we will be specially interested in holomorphic vector bundles, and developing some of Chern's ideas, we will require our fibres to be complex, since it's the natural setting for the theory.

As in many other geometric structures, there is an idea that was formalized and briliantly applied by the French mathematician *Élie Cartan* in the study of differentiable manifolds and Lie groups, the notion of a *local frame* of sections of the bundle of interest. The point of using this frames is that one has through them the ability to express global quantities (like tensorial/spinorial) in terms of local computations. And the notion is very intuitive also from a physical point of view, representing a particular observer in the physical situation of interest. In the setting we are, these frames will have to be compatible with our complex structures.

Thus we want to understand how do this frames behave and transform in our setting. Hence suppose that we have a vector bundle (recall that we take it over $\mathbb{C}$)

$$\pi \colon E \to X,$$

of rank r.

Definition 3.1.1. We say that $f = (e_1, ..., e_n)$ is a *frame* at the point $x \in X$ if we can find a neighborhood U containing the point x and sections of the bundle $e_1, ..., e_j \in \mathcal{E}(U, E)$ that are linearly independent at every point in U.

Now the point of frames is not only to "have" them, but also to understand how do transformation between frames transforms our geometric/physical structures of interest. We have to be able to relate (analytically) one observer (frame) to any other. This is done as follows, suppose that we have a frame $f = (e_1, ..., e_n)$ in a certain neighborhood U of a point $x \in X$. Now if we have a differentiable mapping $\sigma \colon U \to GL_r(\mathbb{C})$. Then we can act with the map σ on frames defined on the open set U in a natural way (just matrix multiplication)

$$f \to f\sigma,$$

where this stands for

$$(f\sigma)(x) = (\sum_{j=1}^{n} \sigma_{j1}(x)e_j(x), ..., \sum_{j=1}^{n} \sigma_{jr}(x)e_j(x)),$$

for every $x \in U$. It is clear (since $\det(\sigma) \neq 0$)) that the new set of sections that we get by acting with σ is another frame, due to the fact that frames are encoded in the independence of the sections, and that is controlled by the determinant, that will not go to zero. Hence we have made a *change of frame* $f \to f\sigma$ (for the frame defined in the open set U). More importantly one has to note here that not also one can act with

$$\sigma \colon U \to GL_r(\mathbb{C})$$

on the set of all possible frames f on the open sets U. But this action is furthermore transitive! This is because given any two frames $f = (e_1, ..., e_r), f' = (e'_1, ..., e'_r)$ in an open set U, one can always find a change of frame σ that turns one into the other, meaning $f' = f\sigma$. This structure can be put into a more geometric structure by introducing the notion of the Frame Bundle, a fiber bundle whose sheaf of sections over open sets is the set of all possible frames. For more details on this construction, the reader is referred to any book on differential geometry, for instance [59].

By means of this local frames, we will be able to find (and do local computations with) local representations for the differential geometric objects that will appear. We will start by using this frames on a geometric object that we have used before, sections of vector bundles. Let $\pi \colon E \to X$ be a vector bundle, and suppose that for an open set $U \subset X$ we have a section $\phi \in \mathcal{E}(U, E)$. Here note that one might think about taking a frame f for this vector bundle on the open set U, but here one has to be careful since the frame need not exist for every open set U (for instance they may fail to exist global sections, or other topological restrictions). One can get around this just by reducing the open set U to be sufficiently small for one to have a frame f in it (since we can do this for small neighborhoods around points, due to the bundle definition), thus we assume that for the open set U (restricted if necessary) we can find a frame f.

With this frame, we can express locally the sections of our bundle by just expanding them in this base. In fact frames are no more than the differential-geometric notion that is analogue to taking a basis in Linear Algebra (here just we have Linear Algebra in movement). Thus we write

$$\phi(s) = \sum_{j=1}^{r} \phi^j[f](s) e_j(s),$$

for every $s \in U$. Where now the $\phi^j[f] \in \mathcal{E}(U)$ are the corresponding uniquely determined functions given by expanding the vector in the basis given by the frame. With this functions, we get a mapping

$$\begin{aligned} l^f \colon \mathcal{E}(U, E) &\to \mathcal{E}(U)^r \simeq \mathcal{E}(U, U \times \mathcal{C}^r) \\ \phi \to l^f(\phi) \colon &= \phi(f) = \begin{bmatrix} \phi^1(f) \\ \vdots \\ \phi^r(f) \end{bmatrix}, \end{aligned}$$

where the components $\phi^j(f)$ are just the components of the section in the frame f.

Now, as we said before, we want to know what happens to this expressions once we change our frame. Hence suppose that we have a change of frame

$$\sigma\colon U \to GL_r(\mathbb{C}).$$

Then by computation one checks that, for our frame f and section ϕ

$$\phi^j(f\sigma) = \sum_{i=1}^{r} \sigma_{ji}^{-1}\phi^i(f),$$

which at the level of the action just reads

$$\phi(f\sigma) = \sigma^{-1}\phi(f),$$

or equivalently

$$\sigma(f\sigma) = \sigma(f),$$

with all products just being matrix multiplications at a certain point $x \in U$, the open set where our frames are defined. Hence we have seen how to obtain a local vector representation for a given section $\phi \in \mathcal{E}(U,E)$ and how this local representation changes under a change of frame for the vector bundle $\pi\colon E \to X$.

If we have even more structure, meaning that the bundle E is a holomorphic vector bundle, one can thus also consider frames $f = (e_1, ..., e_j)$ that are given by holomorphic sections, meaning that $e_j \in \mathcal{O}(U,E)$, and that satisfy $e_1 \wedge ... \wedge e_r(x) \neq 0$ for $x \in U$. This frames, that will be bether adapted to the study of holomorphic-like objects, will be called *holomorphic frames*. If one wants to deal with this holomorphic frames, one has to restrict appropriately the change of frames that may affect them, to deal with this, one restricts attention in this setting to *holomorphic changes of frame*, meaning that now the maps $\sigma\colon U \to GL_k(\mathbb{C})$, that define changes of frames, are required to be holomorphic.

So, if we choose a holomorphic frame f, we get a vector representation

$$\mathcal{O}(U,E) \xrightarrow{l^f} \mathcal{O}(U)^r,$$

that is just given by finding the local expression on the frame

$$\phi \to l^f(\phi) = \phi^j[f],$$

and the same transformation rule that we saw before for general change of frames still holds in the holomorphic setting.

Now that we have introduced this machinery, it is useful to remember what were some of the fundamental concepts that helped develop differential geometry in the real setting and "upgrade" them to this complex geometry. The main concepts that we will deal with are connections, curvatures and metrics, which are fundamental objects if one wants to describe the geometry of spaces and the dynamics inside them. We will define this objects appropriately in this complex setting and give some examples later on. We begin by the fundamental notion of a metric, but since we are in the complex case, metrics will need to be *Hermitian*.

Definition 3.1.2. Let $\pi\colon E \to X$ be a vector bundle. A *Hermitian metric* h on the vector bundle E is an assignment of a Hermitian inner product

$$< \cdot, \cdot >_x \colon E_x \times E_x \to \mathbb{C},$$

to every fibre E_x. Since we are in the smooth category, we need to require this metric h to behave "smoothly". This just amounts to imposing that for any open set $U \subset X$ and sections $\phi, \psi \in \mathcal{E}(U, E)$, the function

$$< \phi, \psi >\colon U \to \mathbb{C}$$
$$< \phi, \psi > (x) =< \phi(x), \psi(x) >_x$$

is a smooth $\mathcal{C}^\infty$ function.

Definition 3.1.3. If a vector bundle $\pi\colon E \to X$ can be equipped with the structure of a Hermitian metric h, then E is called a *Hermitian vector bundle*.

Now note that we can try to express this new geometric object, the metric, locally via our frames. For a Hermitian vector bundle $E \to X$ with metric h and a frame $f = (e_1, ..., e_r)$ on a certain open set $U \subset X$. Then we can write some functions

$$h(f)_{ij} = h(e_i, e_j) =< e_i, e_j >,$$

and denote by $h(f) = (h(f)_{ij})$ denote the $r \times r$ matrix of $\mathcal{C}^\infty$ functions, where $r = rankE$. Due to the fact that h is hermitian at every fibre, this implies algebraically that $h(f)$ is a positive definite Hermitian symmetric matrix, this meaning that it is invertible and

$$h(f)_{ij} = \overline{h(f)_{ji}},$$

which one can see as the complex analog of the symmetry one has in Riemannian metrics in the real case.

What we have obtained is a (local) representation for the hermitian metric h with respect to a certain frame f. Now, for any pair of sections $\phi, \psi \in \mathcal{E}(U, E)$, we can write the local computations (in the frame f) as follows:

$$\begin{aligned} h(\phi, \psi) &= h(\sum_\lambda \phi^\lambda(f) e_\lambda, \sum_\mu \psi^\mu(f) e_\mu) \\ &= \sum_{\lambda,\mu} \overline{\psi_\mu(f)} h_{\mu\lambda}(f) \phi^\lambda(f) \\ h(\phi, \psi) &= \psi^\mathsf{T} h(f) \phi(f), \end{aligned}$$

where the last equation just packs the whole thing as a matrix equation, that is handy for computations, A^{T} just stands for the hermitian conjugate of a matrix (taking transpose and complex conjugation) of a matrix.

Now, we want to know how do this objects (Hermitian metrics) behave under changes of frames. So suppose that we have $\sigma\colon U \to GL_r(\mathbb{C})$ a change of frame, then the following holds

$$h(f\sigma) = \sigma^{\mathsf{T}} h(f)\sigma,$$

and this is the *transformation law* for the local representations of the Hermitian metrics. We can deal locally with this structures just by usual linear algebra.

The natural question to arise here is, can we always use metrics when studying complex vector bundles? And the question is encouraging for us since the answer turns out to be affirmative, and we can find Hermitian metrics on any vector bundle $E \to X$.

Theorem 3.1.1. *Every complex vector bundle (of finite rank) $\pi\colon E \to X$ admits a Hermitian metric.*

The proof can be seen in (Chap. III Theorem 1.2) [1].

The next object that we want to study, since we know that they are fundamental to the study of the geometry of the manifold (for instance in the real case), are *differential forms with coefficients on a vector bundle E.*

Definition 3.1.4. Let $\pi\colon E \to X$ be a vector bundle. We will call the sections of the sheaf

$$\mathcal{E}^p(X, E)\colon = \mathcal{E}(X, \bigwedge^p T^*X \otimes_C E),$$

differential forms of degree p on X with coefficients in the bundle E.

Our objective is now to relate this definition of differential forms to the sheaf-theoretic one involving tensor products **over the structure sheaf**, since as we emphasized when we were talking about sheaves, the structure sheaf is to be thought of as a kind of fundamental object in the geometry. To do this we first need a technical lemma.

Lemma 3.1.1. *If we consider E and E' to be two vector bundles over X, then we have an isomorphism*

$$\Phi\colon \mathcal{E}(E) \otimes \mathcal{E}(E') \to \mathcal{E}(E \otimes E').$$

By specializing this case to our situation we obtain:

Corollary. If $\pi\colon E \to X$ is a vector bundle over X. Then we have an isomorphism

$$\mathcal{E}^p \otimes \mathcal{E}(E) \simeq \mathcal{E}^p,$$

which is just given by the isomorphism in the previous lemma.

Now we denote the image of $\alpha \otimes \phi$ under the isomorphism given by the corollary by $\alpha \cdot \phi \in \mathcal{E}^p(X, E)$. In this assignment, $\alpha \in \mathcal{E}^p$ is a differential form and $\phi \in \mathcal{E}(X, E)$ is a section of the bundle. Now we want to see how all this looks locally, to be able to do computations.

So suppose that we have a local frame $f = (e_1, ..., e_r)$ for the bundle $E \to X$ over a certain open set $U \subset X$. Then we have the local representation for $\xi \in \mathcal{E}^p(U, E)$ defined by

$$\xi = \sum_{j=1}^{r} \xi^j e_j.$$

More precisely, for $x \in U$ and $(\omega_1, ..., \omega_s)$ a frame for $\wedge^p T^*X \otimes \mathbb{C}$ around the point x. We can then write locally

$$\xi(x) = \sum_{i,j} \phi_{ij}(x)\omega(x)_j \otimes e_i(x),$$

where ϕ_{ij} are uniquely determined smooth functions defined near x. We can consider

$$\xi^p = \sum_{j} \phi_{jp}\omega_j,$$

so as to view them as $E-$valued forms, a perspective that was emphasized by Élie Cartan.

We want to study how this objects change. What we need is the introduction of the notion of *connection* adapted to this complex setting:

Definition 3.1.5. Consider $\pi \colon E \to X$ to be a vector bundle. A *connection* ∇ on the bundle $E \to X$ is defined as a $\mathbb{C}-$linear mapping

$$\nabla \colon \mathcal{E}(X, E) \to \mathcal{E}^1(X, E),$$

that satisfies a "Leibniz rule" for every $\alpha \in \mathcal{E}(X)$ and $\psi \in \mathcal{E}(X, E)$, namely

$$\nabla(\alpha\psi) = d(\alpha) \cdot \psi + \alpha\nabla(\psi).$$

It is useful to look at this in a concrete easy example, just taking the trivial bundle over a manifold.

Example 3.1. Consider the trivial line bundle $X \times \mathbb{C} \to X$. We note that we can take just the ordinary exterior differentiation operator

$$d \colon \mathcal{E}(X) \to \mathcal{E}^1(X)$$

as a connection on the trivial bundle E, since the ordinary (complex) exterior derivative satisfies the requirements (basically the Leibniz rule). This tells us that connections in arbitrary bundles are a generalization of the exterior differential operator to E-valued differential forms, and we will later try and push this operator (the connection) to act on higher order forms.

As before with the metric, we want to know how to work locally (and compute) with connections. To do this we take a frame $f = (e_1, ..., e_r)$ for a vector bundle $E \to X$. We suppose that the bundle is equipped with a certain connection ∇. From this we define the local *connection matrix* $\theta(\nabla, f)$, that will depend on the frame we are taking and on the connection by

$$\theta(\nabla, f) = (\theta_{\mu\nu}(D, f)), \theta_{\mu\nu}(\nabla, f) \in \mathcal{E}^1(U),$$

and satisfying

$$\nabla e_\lambda = \sum_{\mu=1}^{r} \theta_{\mu\lambda}(\nabla, f) \cdot e_\mu.$$

And now, given this matrix coefficients, how does one compute with the connection? Just by applying the "Leibniz rule" on the given frame f, and thus obtaining:

$$D\xi(f) = d\xi(f) + \theta(f)\xi(f) = [d + \theta(f)]\xi(f),$$

and we think of $D = d + \theta(f)$ as being a linear operator acting on vector-valued functions (locally).

If we make the same construction in the real setting we have, for a given frame $\omega = (\omega_1, ..., \omega_n)$ for the cotangent bundle T^*X over U we have

$$\theta_{\mu\nu} = \sum_k \Gamma^\mu_{\nu k} \omega_k,$$

with $\Gamma^\mu_{\nu k} \in \mathcal{E}(U)$, and these are just the classical (Schwarz-)Christoffel symbols associated with a connection in the real setting.

Recalling what we have done, we note that we have introduced metrics and connections into our geometric description. There is one element that we talked about before that is missing, and that is indeeed the *curvature*.

Let $E \to X$ be a vector bundle with connection ∇. We denote by $Hom(E, E) \simeq E^* \otimes E$ the vector bundle with fibres $Hom(E_x, E_x)$. We want to see that the connection ∇ induces in a natural manner an element

$$\Theta_E(\nabla) \in \mathcal{E}^2(X, Hom(E, E)),$$

that we will call the *curvature tensor*.

Definition 3.1.6. Let $\pi \colon E \to X$ be a vector bundle, with a certain connection ∇. Following the notation used previously we denote by $\theta(\nabla, f)$ the associated connection matrix for a certain frame $f = (e_1, ..., e_r)$. Now we define the following object:

$$\Theta(\nabla, f) = d\theta(f) + \theta(f) \wedge \theta(f),$$

and this is just an $r \times r$ (r being the rank of the bundle E) matrix of 2-forms, meaning:

$$\Theta_{\mu\nu} = d\theta_{\mu\nu} + \sum_{\lambda} \theta_{\mu\lambda} \wedge \theta_{\lambda\nu}.$$

The matrix $\Theta(\nabla, f)$ is called the *curvature matrix* for the connection ∇ in the frame f, meaning that it can be computed from the connection matrix $\theta(\nabla, f)$.

Now that we have defined this curvature matrix Θ, we want to see how does it transform under changes of frame.

Lemma 3.1.2. *Suppose that we are in the geometric situation above, with a bundle $E \to X$ and a connection ∇. And that we are considering the local quantities $\theta(f)$ and $\Theta(f)$ as defined above. Then we have*

- $d\sigma + \theta(f)\sigma = \sigma\theta(f\sigma)$.
- $\Theta(f\sigma) = \sigma^{-1}\Theta(f)\sigma$.

The proof just follows by a direct computation.

Now that we know how the local representation of Θ varies under an arbitrary change of frame, we can try to relate two operators. First the operator that we have defined as *curvature* ($\Theta_{\mu\nu}$) and the operator $d + \theta(f)$. It turns out that they have a very explicit relationship, that we see in the next lemma.

Lemma 3.1.3. *The following equation is satisfied*

$$[d + \theta(f)][d + \theta(f)]\phi(f) = \Theta(f)\phi(f),$$

for an arbitrary frame $f = (e_1, ..., e_r)$.

Proof. Note that we will compute with the operators directly (reducing the notational dependence on the frame f). And by direct computation we obtain:

$$\begin{aligned}
(d+\theta)(d+\theta)\phi &= d^2\phi + \theta \cdot d\phi + d(\theta \cdot \phi) + (\theta \wedge \theta) \cdot \phi \\
&= \theta \cdot d\phi + d\theta \cdot \phi - \theta \cdot d\phi + (\theta \wedge \theta) \cdot \phi \\
&= d\theta \cdot \phi + (\theta \wedge \theta) \cdot \phi \\
&= (d + \theta \wedge \theta) \cdot \phi \\
&= \Theta \cdot \phi.
\end{aligned}$$

□

This proof just allows us to note the importance of having related abstract operations and equations at the theoretical level with "down-to-earth" matrix operations and equations (tensor calculus).

We thus make the following definition of the curvature (as a global notion).

Definition 3.1.7. Let $\pi\colon E \to X$ be a vector bundle with a connection ∇. Then the *curvature* $\Theta_E(\nabla)$ is defined to be the global element $\Theta \in \mathcal{E}^2(X, Hom(E, E))$ such that the $\mathbb{C}-$linear mapping defined by $Theta$, namely

$$\Theta\colon \mathcal{E}(X, E) \to \mathcal{E}^2(X, E),$$

just looks locally like a curvature matrix, meaning that with respect to a frame f, the mapping is just

$$\Theta(f) = \Theta(\nabla, f) = d\theta(f) + \theta(f) \wedge \theta(f).$$

We can relate the curvature to the exterior derivative, and in fact the curvature will turn out to be the measure of the failure to be exact of the sequence defined by the exterior covariant derivative. More precisely,

Lemma 3.1.4. $D^2 = \Theta$ *as an operator mapping*

$$\mathcal{E}^p(X, E) \to \mathcal{E}^{p+2}(X, E).$$

This just follows from a direct computation.

Recall that in real differential geometry, the Riemann curvature tensor satisfies some special identities that are known as *Bianchi identities*. In this setting, these identities are of the following form.

Lemma 3.1.5. $d\Theta(f) = [\Theta(f), \theta(f)]$

Proof. As usual, this follows by a direct computation, we will show this proof to see how these and other proofs in this context are made. If we let $\theta = \theta(f)$ and $\Theta = \Theta(f)$ then

$$\Theta = d\theta + \theta \wedge \theta,$$

thus we have that

$$d\Theta = d^2\theta + d\theta \wedge \theta - \theta \wedge d\theta = d\theta \wedge \theta - \theta \wedge d\theta.$$

We also have

$$[\Theta, \theta] = [d\theta + \theta \wedge \theta, \theta] = d\theta \wedge \theta - \theta \wedge d\theta.$$

Note that both computations give the same result, thus proving the lemma. $\square$

This is a clear example of what we mean by a proof which just "follows by direct computation".

A connection D on a bundle E is said to be *compatible* with a Hermitian metric h on E if

$$d < \tau, \sigma > = < D\tau, \sigma > + < \tau, D\sigma > .$$

And as happens in the real case, for every Hermitian vector bundle, we can find connections which are compatible with it. One should note that this connection is not unique (as in the real case). If one wants uniqueness one has to require some additional properties, and that depends on the geometry. In the real case one imposes the connection to be torsionless (Levi-Civita), and in the Hermitian holomorphic case we will see in the next section how we obtain this uniqueness.

3.2 Connections and curvature on Hermitian Holomorphic vector bundles.

In this section we will want to restrict our attention to holomorphic-like structures, and we will see how the hermitian structures that we defined and studied in the previous section look like in this holomorphic setting. We make the following definition, of the structure that we will want to understand better in this section.

Definition 3.2.1. Let $\pi\colon E \to X$ be a holomorphic vector bundle over a complex manifold X. Now, if we have that E, as a differentiable bundle, can be equipped with the structure of a differentiable Hermitian metric h, we will call the bundle E a *Hermitian holomorphic vector bundle.*

This kind of structures will allow us to see the objects of the previous sections through the lenses of complex analysis (provided that we restruct our attention to holomorphic bundles). For instance, recall that since X is a complex manifold, we have the following decomposition for it's differential forms

$$\mathcal{E}^*(E) = \bigoplus_r \mathcal{E}^r(X) = \bigoplus_{p,q} \mathcal{E}^{p,q}(E),$$

where the components $\mathcal{E}^{p,q}$ of the decomposition induced by the complex structure are just

$$\mathcal{E}^{p,q}(E) = \mathcal{E}^{p,q}_X \otimes_{\mathcal{E}_X} \mathcal{E}.$$

Now suppose that we have a connection in our bundle $E \to X$, meaning that we have a map

$$\nabla\colon \mathcal{E}^{X,E} \to \mathcal{E}^1(X,E) = \mathcal{E}^{1,0}(X,E) \oplus \mathcal{E}^{0,1}(X,E).$$

Now note that due to the splitting on the image, the connection ∇ splits naturally into two operators $\nabla = \nabla^{1,0} + \nabla^{0,1}$ (just by composing with the appropriate projections), where we have

$$\nabla^{1,0}\colon \mathcal{E}^{X,E} \to \mathcal{E}^{1,0}(X,E)$$
$$\nabla^{0,1}\colon \mathcal{E}^{X,E} \to \mathcal{E}^{0,1}(X,E)$$

And we furthermore have a connection in our bundle E, meaning that we have a Hermitian holomorphic vector bundle, we can characterize the relationship between connections and hermitian metrics, and we give this in the form of a theorem.

Theorem 3.2.1. *Consider that h is a Hermitian metric on a certain holomorphic vector bundle $E \to X$. Then h induces canonically a connection (∇_h) on the bundle E satisfying, for every open set $U \subset X$,*

- *For every two sections $\phi, \psi \in \mathcal{E}(U, E)$ we have*

$$d(h(\phi, \psi)) = h(\nabla\phi, \psi) + h(\phi, \nabla\psi),$$

  this just means that the connection ∇ is compatible with the metric.

- *If we take a* ***holomorphic*** *section $\phi \in \mathcal{O}(U, E)$ then we get $\nabla^{0,1}\phi = 0$.*

The proof of this theorem can be seen in (Chap. III Theorem 2.1) [1].

The important thing to realise with this theorem, is that *holomorphic frames* (meaning that they are given by holomorphic sections of the bundle) make our local expresions easier to handle. For instance, if we have a holomorphic frame $f = (e_1, ..., e_r); e_j \in \mathcal{O}(U, E)$, then the connection matrix $\theta(f)$ can be easily found in terms of the metric as

$$\theta(f) = h(f)^{-1}\partial h(f).$$

Moreover, the decomposition of the connection given by the complex structure ($\nabla = \nabla^{1,0} + \nabla^{0,1}$) is, in a holomorphic frame f, just given by the following simplified expression

$$\nabla^{1,0} = \partial + \theta(f)$$
$$\nabla^{0,1} = \overline{\partial}.$$

From this expression we can deduce the following result (remember that we are restricting ourselves to holomorphic frames!).

Lemma 3.2.1. *Let $\pi \colon E \to X$ be a Hermitian holomorphic vector bundle with metric h. Denote by ∇ the induced connection on E (given by the previous theorem). Now we consider a holomorphic frame $f = (e_1, ..., e_r)$, and we denote by $\theta(f)$ and $\Theta(f)$ the induced connection and curvature matrices respectively, that are determined by the connection ∇ and the frame f. Then we have*

- *$\theta(f)$ is of type $(1,0)$, and $\partial\theta(f) = -\theta(f) \wedge \theta(f)$.*
- *$\Theta(f) = \overline{\partial}\theta(f)$, and $\Theta(f)$ is of type $(1,1)$.*
- *$\overline{\partial}\Theta(f) = 0$, and $\partial\Theta(f) = [\Theta(f), \theta(f)]$.*

This just follows from a direct computation by using the relationship between the metric and the connection matrix.

3.3 Chern Classes associated to differentiable vector bundles

In this section we will introduce the notion of *Chern classes*, which are certain invariants that are encoded as differential forms. This invariants are also of use in physics, for instance in the quantization of Dirac's Monopole. Chern classes turn out to be actually the obstruction to admitting a certain number of global sections.

To begin, we first need some notions from multilinear algebra. If we denote by $\mathcal{M}_r$ the set of $r \times r$ matrices with complex entries. Then a $k-$linear form

$$\overline{\varphi}\colon \mathcal{M}_r \times \cdots \times \mathcal{M}_r \to \mathbb{C}$$

is said to be *invariant* if for every element $g \in GL(r, \mathbb{C})$, and $A_j \in \mathcal{M}_r$ we have

$$\overline{\varphi}(gA_1g^{-1}, ..., gA_kg^{-1}) = \overline{\varphi}(A_1, ..., A_k).$$

An invariant $k-$linear form clearly induces a map

$$\varphi\colon \mathcal{M}_r \to \mathbb{C},$$

which will be a homogeneous invariant polynomial of degree k. And one can also go in the reverse direction by looking at the symmetric algebra.

The natural example of this type of object is the determinant, and this will be of use later for defining Chern classes.

We consider the homogeneous polynomials associated to the determinant in the following way:

$$det(Id + A) = \sum_{k=0}^{r} \Phi_k(A),$$

where each Φ_k is an invariant homogeneous polynomial of degree k. What we do now is extend the action to arbitrary elements of $\mathcal{E}^*(Hom(E, E))$ in the natural way. And thus we have

$$\phi_x\colon \mathcal{E}^p(X, Hom(E, E)) \times \cdots \times \mathcal{E}^p(X, Hom(E, E)) \to \mathcal{E}^{pk}(X),$$

we denote by ϕ_X the restriction to the diagonal.

Suppose that we have a connection D

$$D\colon \mathcal{E}(X, E) \to \mathcal{E}^1(X, E),$$

on the bundle $E \to X$. Then we have the associated curvature $\Theta_E(D) = D^2$. And thus, if $\phi \in I_k(\mathcal{M}_r)$, then $\phi_X(\Theta_E(D))$ is a global $2k-$form on the base manifold X. And we have a fundamental result that is what justifies the subsequent development. This result is due to A. Weil, and a proof can be seen in [1].

Theorem 3.3.1. *Under the conditions above, we have the following:*

1. *$\phi_X(\Theta_E(D))$ is a closed form.*
2. *The image in the de Rham group $H^{2k}(X,\mathbb{C})$ is independent of the connection D chosen.*

With this result, we can now define what Chern classes of a differentiable vector bundle are.

Definition 3.3.1. Let $E \to X$ be a differentiable vector bundle with a connection D. Then the kth *Chern form of E relative to the connection D* is defined as

$$c_k(E,D) \colon = (\Phi_k)_X(\frac{i}{2\pi}\Theta_E(D)) \in \mathcal{E}^{2k}(X).$$

The *(total) Chern form of E relative to D* is then defined as the sum

$$c(E,D) = \sum_{k=0}^{r} c_k(E,D),$$

r being just the rank of the bundle E.

The kth *Chern class* of the vector bundle E, that will be denoted by $c_k(E)$ is the cohomology class of the form $c_k(E,D) \in H^{2k}(X,\mathbb{C})$. The *total Chern class* of E, denoted by $c(E)$ is the cohomology class of the form $c(E,D) \in H^*(X,\mathbb{C})$, meaning

$$c(E) = \sum_k c_k(E).$$

These objects where defined by the chinese mathematician Shiing-Shen Chern. From the theorem above, it follows that Chern classes are well defined objects (since the cohomology class does not depend on the connection). And thus Chern classes are topological cohomology classes in the base space of the bundle E. In fact one can see that Chern classes are real differential forms (Chap III. Proposition 3.5) in [1]. In fact they are integral!

One of the most important properties of this classes, and that makes them useful, are their functoriality properties, this are sumarized in the following theorem, whose proof the reader can check in [1].

Theorem 3.3.2. *Suppose that E, E' are differentiable $\mathbb{C}-$vector bundles over a differentiable manifold X. Then we have:*

1. *For any smooth map $f \colon Y \to X$ (Y being another smooth manifold) we have*

$$c(f^*E) = f^*c(E).$$

2. *$c(E \oplus E') = c(E)c(E')$, where the product is in the cohomology ring.*

3. *$c(E)$ depends only on the isomorphism class of the bundle E.*

4. *If E^* denotes the dual bundle of E, then we have*

$$c_j(E^*) = (-1)^j c_j(E).$$

We said that Chern classes measure obstructions to finding global sections, and this is the content of the following result, its proof can be seen in [1].

Theorem 3.3.3. *If $E \to X$ is a differentiable bundle of rank r, then we have*

1. *$c_0(E) = 1$.*
2. *If $E \cong X \times \mathbb{C}^r$ (is trivial) then $c_j(E) = 0$ for $j = 1, ..., r$, meaning that $c(E) = 1$.*
3. *If $E \cong E' \oplus T_s$, where T_s is a trivial vector bundle of rank r, then $c_j(E) = 0$ for $j = r - s + 1, ..., r$.*

3.4 Hermitian Exterior Algebra on a Hermitian vector space

In this section we will restrict our attention to the study of **compact** manifolds, where we will apply the notions of the previous sections that will be needed. We will start with some algebraic notions and then apply them to geometry.

We start by recalling the notion of the *Hodge operator*. If V is a real finite-dimensional vector space equipped with an inner product $<,>$, we know that it can be naturally be extended to the exterior algebra $\bigwedge V$. The hodge operator is the unique operator satisfying

$$\alpha \wedge *\beta =< \alpha, \beta > vol,$$

for arbitrary $\alpha, \beta \in \wedge^p V$ and vol being the usual volume form.

This structure can be naturally extended to the complex setting, since one can define an inner product in $\wedge^p V \otimes \mathbb{C}$ as $< \alpha, \beta >: \; = \sum'_{|I|=p} \alpha_I \overline{\beta}_I$. In the case that α, β are real, we recover the real inner product, thus we use the same symbol for both. One can extend the real hodge operator $*$ to this complex setting by requiring complex linearity, thus obtaining

$$\alpha \wedge *\overline{\beta} =< \alpha, \beta > vol.$$

For a complex vector space E of complex dimension n we consider F to be the complex vector space of complex-valued real-linear mappings of E to $\mathbb{C}$, which has complex dimension $2n$. The exterior algebra $\wedge F$ comes equipped with the bigrading structure

$$\wedge F = \bigoplus_{r=0}^{2n} \bigoplus \wedge^{p,q} F.$$

Now suppose that our complex vector space E is equipped with a Hermitian inner product $<,>$. Then, in a basis of the type $\{z_1, ..., z_n, \overline{z}_1, ..., \overline{z}_n\}$, for the respective dual space, one can write the inner product

$$< u, v >= h(u, v),$$

as

$$h = \sum_{\mu,\nu} h_{\mu\nu} z_\mu \otimes \overline{z}_\nu,$$

where $(h_{\mu\nu})$ is a positive definite Hermitian symmetric matrix. Now the inner product h can be naturally decomposed into two parts

$$h = S + iA,$$

where S and A are real bilinear forms. S is a symmetric positive-definite bilinear form representing the Euclidean inner product on the underlying real vector space induced by the Hermitian metric h. As for A, one can see that

$$A = \frac{1}{2i} \sum_{\mu,\nu} h_{\mu\nu}(z_\mu \otimes \overline{z}_\nu - \overline{z}_\nu \otimes z_\mu) = -i \sum_{\mu,\nu} h_{\mu\nu} z_\mu \wedge \overline{z}_\nu.$$

With this in mind we define

$$\Omega = \frac{i}{2} \sum_{\mu,\nu} h_{\mu\nu} z_\mu \wedge \overline{z}_\nu,$$

the *fundamental* $2-$*form* associated to the Hermitian metric h. Clearly we have

$$\Omega = -\frac{1}{2} A = -\frac{1}{2} Im(h),$$

and thus we have that

$$h = S - 2i\Omega,$$

and Ω is nothing more than a real $2-$form. And the associated volume form is just

$$vol = \frac{1}{n!}\Omega^n.$$

Now that we have the form Ω (the fundamental form associated with the Hermitian structure), we define the linear map

$$L\colon \wedge F \to \wedge F,$$

by $L(v) = \Omega \wedge v$. Clearly $L\colon \wedge^p F \to \wedge^{p+2} F$, and moreover

$$L\colon \wedge^{p,q} F \to \wedge^{p+1,q+1} F,$$

and since Ω is a real form, L is a real operator. With respect to the inner product induced by the Hodge operator L has a Hermitian adjoint

$$L^*\colon \wedge^p F \to \wedge^{p-2} F,$$

and in fact $L^* = w * L*$, $\quad w = \sum (-1)^r \pi_r$ where π_r stands for the projection onto the forms of homogeneous degree r.

3.5 Harmonic Theory in the Compact setting

In this section we will devote to the study of harmonic theory in the case of a compact (differentiable or complex) Riemannian/Hermitian manifolds.

The main general theorem that justifies why Harmonic Theory is useful was proven first by Hodge for the case of the de Rham complex. This theorem was later generalized to the setting of *elliptic complexes*, as is stated in [1] (Chap. IV Theorem 5.2). It basically tells us that every cohomology class has a unique harmonic representative, and that the space of harmonic sections is isomorphic to the cohomology. Hence telling us that there is a direct relationship between the topology of the manifold and the kernel of a certain differential operator, in this case the laplacian associated to a certain elliptic complex.

Suppose that X is a compact oriented $d-$dimensional Riemannian manifold. The orientation and the metric define the Hodge operator, which is a smooth bundle map

$$*\colon \wedge^p T^*X \to \wedge^{d-p} T^*X,$$

and can also be extended to the complex case as we will see later.

If we denote the metric by $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$ (using the usual Einstein summation convention of summing repeated indices). And we denote the inverse by $g^{\mu\nu}$ then for a given form

$$\alpha = \sum \alpha_{i_1,...,i_p} dx^{i_1} \wedge ... \wedge dx^{i_p},$$

then we can write the hodge operator in local coordinates as

$$(*\alpha) = \sum \alpha_{j_1,...,j_{d-p}} dx^{j_1} \wedge ... \wedge dx^{j_{d-p}},$$

where

$$(*\alpha)_{j_1,...,j_{d-p}} = \pm\sqrt{det(g)}\alpha^{i_1,...,i_p},$$

where $\{i_1, ..., i_p, j_1, ..., j_{d-p}\} = \{1, ..., d\}$, and the sign depends on the parity of the permutation. In particular we can write the volume form as

$$(*1) = \sqrt{det(g)} dx^1 \wedge ... \wedge dx^d.$$

Using the map $*$, one can define

$$(\phi, \psi) = \int_X \phi \wedge *\overline{\psi},$$

for $\phi, \psi \in \mathcal{E}^p(X)$. And we take $(\phi, \psi) = 0$ if $\phi \in \mathcal{E}^p(X), \psi \in \mathcal{E}^q(X)$ with $p \neq q$.

The same definition can be given for noncompact manifolds by considering compactly-supported forms.

One can check that this definition induces a positive definite Hermitian symmetric sesquilinear form on $\mathcal{E}^*(X)$. This inner product is usually called the *Hodge inner product*. If X has the structure of a Hermitian complex manifold, then this inner product can be given with respect to the underlying differentiable structure. In this setting, the usual direct sum decomposition $\mathcal{E}^r(X) = \bigoplus_{p+q=r} \mathcal{E}^{p,q}(X)$ is an orthogonal direct sum decomposition.

Computing the adjoints of several linear operators acting on $\mathcal{E}^*(X)$ will be easier by using the Hodge operator. But first we will modify this operator a bit to make it more convenient for this purpose.

We define, on an oriented Riemannian manifold,

$$\overline{*} \colon \mathcal{E}^*(X) \to \mathcal{E}^*(X),$$

by $\overline{*}(\alpha) = *\overline{\alpha}$. And now $\overline{*}$ is a conjugate-linear isomorphism of bundles,

$$\overline{*} \colon \wedge^p T^*X_c \to \wedge^{m-p} T^*X_c,$$

with $m = dim_{\mathbb{R}} X$. Now suppose that we have a Hermitian complex manifold and a Hermitian vector bundle $E \to X$. We denote by

$$\tau \colon E \to E^*$$

the conjugate-linear bundle isomorphism from E onto its dual bundle E^*. We can define now the analog of the Hodge operator in this setting

$$\overline{*}_E \colon \wedge^p T^*X_c \otimes E \to \wedge^{m-p} T^*X_c \otimes E^*,$$

by

$$\overline{*}_E(\phi \otimes e) = \overline{*}(\phi) \otimes \tau(e).$$

Note that the Hodge inner product on $\mathcal{E}^*(X)$ can be written as

$$(\phi, \psi) = \int_X \phi \wedge \overline{*}\psi,$$

and with the generalization that we made, we can extend this to an inner product over $\mathcal{E}^*(X, E)$ in a similar fashion by

$$(\phi, \psi) = \int_X \phi \wedge \overline{*}_E \psi,$$

for $\phi, \psi \in \mathcal{E}^r(X, E)$.

In this case, the bigrading is also an orthogonal decomposition with respect to this inner product. We will now see various adjoints for certain common geometric situations.

Lemma 3.5.1. *Let X be an oriented compact Riemannian manifold of real dimension m with the associated laplacian $\Delta = dd^* + d^*d$, with d^* being the adjoint associated to the Hodge inner product on $\mathcal{E}^*(X)$. Under this situation we have:*

1. $d^* = (-1)^{m+mp+1}\overline{*}d\overline{*} = (-1)^{m+mp+1} * d*$*, on* $\mathcal{E}^p(X)$.
2. *The laplacian commutes with the hodge operator, meaning* $*\Delta = \Delta*$ *and* $\overline{*}\Delta = \Delta\overline{*}$.

This result is mainly a consequence of Stoke's theorem, the interested reader can check the detailef proof in (Chap. V Proposition 2.3) [1].

For the Hermitian case we have a similar result. Note that $\overline{*}_{E^*}$ is defined analogously to $\overline{*}_E$ by using the inverse $\tau^{-1}\colon E^* \to E$.

Lemma 3.5.2. *Let X be a Hermitian complex manifold and $E \to X$ a Hermitian vector bundle. Then we have*

1. $\overline{\partial}\colon \mathcal{E}^{p,q}(X,E) \to \mathcal{E}^{p,q+1}(X,E)$ *has an adjoint* $\overline{\partial}^*$ *with respect to the Hodge inner product given by*
$$\overline{\partial}^* = -\overline{*}_{E^*}\overline{\partial}overline{*}_E.$$
2. *If we denote by* $\overline{\Box} = \overline{\partial}\overline{\partial}^* + \overline{\partial}\overline{\partial}^*$ *the associated laplacian acting on* $\mathcal{E}^{**}(X,E)$*, then*
$$\overline{\Box}\overline{*}_E = \overline{*}_E\overline{\Box}.$$

The proof is almost the same as the previous one, and also a consequence of Stoke's theorem, the interested reader is referred to (Chap. V Proposition 2.4) in [1].

It should be noted that only the operator $\overline{\partial}$ acts naturally on $\mathcal{E}^{p,q}(X,E)$ for a non-trivial holomorphic bundle E. Neither d nor ∂ can act naturally in this space, since they will not annihilate the transition functions defining the bundle. However, if one is interested only in the scalar case, then there is no problem, and we also have

$$\partial\colon \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q}(X),$$

and with a very similar calculation as the above lemmas, one gets that $\partial^* = -\overline{*}\partial\overline{*}$, and also the associated laplacian $\Box = \partial\partial^* + \partial^*\partial$ also commutes with $\overline{*}$, exactly as the $\overline{\partial}$ operator we saw above.

3.5.1 Representations of the Lie Algebra sl2

We will devote a bit of time now to something that may seem as something completely different from what we were studying. Since our goal in this section is to better understand the group $SL(2,\mathbb{C})$, specially through it's *Lie Algebra*. We will want to find also representations, that will allow us to better understand these abstract algebras and also to apply these representations to our geometric problems. It also turns out that this group is not just of interest for complex geometry, several subgroups of it are also of interest, for instance in number theory, where the *modular group* $SL(2,\mathbb{Z})$ (or some *congruence subgroup* $\Gamma(N)$) is of fundamental importance. We will not devote to this number-theoretical analysis, but there is another place where this group makes an appearance, that we will be more interested in. Briefly, the group $SL(2,\mathbb{C})$ is just the double cover of the group of symmetries of interest for relativity! The group turns out to be just the $Spin(1,3)$ group, being fundamental for understanding thus relativistic particles, following the program that was highly developed and encouraged both by Hermann Weyl and Eugene Wigner of understanding fundamental particles via (certain class of) representations of the group of symmetries of our system. But we will have more to say on the physics later, now we turn to the study of the properties of $\mathfrak{sl}(2,\mathbb{C})$.

In order to study the algebra, we will devote to its **finite-dimensional** complex representation theory for the *Lie Algebra* $\mathfrak{sl}(2,\mathbb{C})$ and later we will apply this results to specific representations that may arise from Hermitian exterior algebras.

But before we start, we need some notions from the theory of representations of Lie Algebras. We start by giving some definitions.

Definition 3.5.1. Given a vector space A we say that it is a *Lie Algebra* if it is equipped with a Lie bracket product

$$[\cdot,\cdot]\colon A\times A\to A$$

which is bilinear, anticommutative and satisfies the *Jacobi identity*

$$[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0,$$

for every $x,y,z\in A$.

Remark. The prototypical example of a Lie Algebra is any algebra of matrices that we equip with the commutator $[A,B]=AB-BA$. But apart from matrices, there are a lot of geometric situations where Lie algebras do appear in a geometric setting, for instance if one considers vector fields endowed with the differential-geometric Lie Bracket of vector field, one obtains a Lie algebra. The same can be done for *Symplectic* or *Hamiltonian* vector fields in the setting of symplectic geometry, since they are both closed under the Bracket operation. From those one can get Lie algebras on the ring of functions, the so-called *Poisson structures.* Another fundamental place where this algebras appear is in the study of Lie Groups (hence the name), since one can associate an algebra to a Lie group in a natural way, and it turns out that one can study many of the properties of the group by looking at its algebra. This program was carried in detail first by the german mathematician Wilhelm Killing, and afterwards by Élie Cartan.

What we will try to do in this section is to apply some of the ideas of Cartan to our particular case of interest, the group $SL(2,\mathbb{C})$ and it's Lie Algebra.

Definition 3.5.2. A *representation* of the Lie Algebra A on a vector space V is just an algebra homomorphism

$$\rho\colon A \to End(V),$$

where $End(E)$ is just the Lie Algebra of endomorphisms of V equipped with the usual commutator

$$[A, B] = AB - BA.$$

If the vector space V is finite dimensional, meaning $dimV = n < \infty$, then call ρ a *finite-dimensional representation* of dimension n. If the vector space V is infinite-dimensional, we say that ρ is an *inifinite-dimensional representation.*

Remark. One can induce, from some algebraic constructions in the category of vector spaces, the appropriate construction for representations. For instance if ρ_1 and ρ_2 are two representations on vector spaces V_1 and V_2, then one can build in a natural manner the representation $\rho_1 \oplus \rho_2$ on the vector space $V_1 \oplus V_2$.

We call two representations $\rho_1\colon A \to End(V_1)$ and $\rho_2\colon A \to End(V_2)$ *equivalent* if we can find an isomorphism $S\colon V_1 \to V_2$ such that $\rho_1 = S^1 \rho_2 S$.

Now that we have the notion of a representation, we want to define a special kind of representation that will turn out to be our "building blocks" for the representations we are interested in, and through the course of the section we will see why.

Definition 3.5.3. A representation $\rho\colon A \to End(V)$ is said to be *irreducible* if there is no proper invariant subset $0 \neq V_0 \subset V$. Meaning that we there is no proper subspace $0 \neq V_0 \subset V$ satisfying

$$\rho(a)V_0 \subset V_0,$$

for every $a \in A$.

Remark. This representations are what correspond to the idea of a fundamental particle in physics, associated to certain special lie groups that are of interest in physics.

We say that a arepresentation ρ is *completely reducible* if it is equivalent to a direct sum of irreducible representations.

For our interest in this section, we will try to understand a concrete Lie Algebra, namely $\mathfrak{sl}(2, \mathbb{C})$. This algebra is just defined as the 2×2 matrices with entries in $\mathbb{C}$ with zero trace. To better understand this algebra we see it as the Lie Algebra associated to a group. More specifically we take the group $SL(2, \mathbb{C})$ (the name is obviously on purpose), which is just the group of 2×2 matrices with entries in $\mathbb{C}$ having determinant 1. One can see the explicit relationship between the groups via the exponential map, which is just defined by the usual series since we are dealing with matrix-groups

$$\begin{aligned} exp\colon \mathfrak{sl}(2, \mathbb{C}) &\to SL(2, \mathbb{C}) \\ X \to &exp\{X\} = e^X = \sum_{j=0}^{\infty} \frac{X^n}{n!}, \end{aligned}$$

this expression is norm convergent (for instance in the $\|\cdot\|_2$ in the space of matrices) for every matrix, thus inducing a well defined element in the space of 2×2 complex matrices. Now to see that we have a well defined map (meaning that the exponential is in $SL(2, \mathbb{C})$, we just use the classical Jacobi formula, which gives us that the general identity

$$det(exp(A)) = det(e^A) = e^{Tr(A)},$$

holds for any matrix, where the exponential on the last equality is just the holomorphic function e^z. From this we can infere that, since our algebra elements are traceless, this just means that their respective exponentials will have determinant 1, and this is just the condition defining the image $SL(2, \mathbb{C})$, hence the map exp is well defined.

Now, we want to see how does the algebra structure relate to information on the group, and what we observe is that we, in a sense, "see" the algebra when we look at the group up to order 1 and 2. This just amounts to saying that, for $t \in \mathbb{C}$ and $X, Y \in \mathfrak{sl}(2, \mathbb{C})$, we have the following expansions

$$\begin{aligned} e^{tX} &= \mathbb{1} + tX + O(|t|^2) \\ e^{tX}e^{tY} &= \mathbb{1} + t(X+Y) + O(|t|^2) \\ e^{tX}e^{tY}e^{-tX}e^{-tY} &= \mathbb{1} + t^2([X,Y]) + O(|t|^3), \end{aligned}$$

where $\mathbb{1}$ just stands for the identity matrix. Hence our view that the group encodes locally the algebra is clear, and in fact in an abstract setting it is precisely the definition of the lie algebra, the tangent space, with the bracket constructed appropriately.

Now we turn our attention to a special subgroup of our group $SL(2, \mathbb{C})$. The group of unitary matrices of determinant 1, the group that is usually denoted by $SU(2)$. To this subgroup of $SL(2, \mathbb{C}$ it has to correspond a certain subalgebra in $\mathfrak{sl}(2, \mathbb{C})$, that we will denote by $\mathfrak{su}(2)$, and that turns out to be formed by skew-Hermitian matrices traceless matrices, meaning $X + X^* = 0, Tr(X) = 0$. This is just deduced from the expansions that we derived before since, if we require that the exponentials e^X belong to the subgroup $SU(2)$ we are imposing the condition that they are unitary

$$(e^X)^\dagger e^X = \mathbb{1}.$$

Now we have to use that, for $t \in \mathbb{C}$, $(e^{tX})^\dagger = e^{\bar{t}X^\dagger}$. This is just seen by expanding and using the continuity of the (hermitian) adjoint. Using this we obtain, by taking $t \in \mathbb{R}$

$$\mathbb{1} = (e^{tX})^\dagger e^{tX} = exp\{tX^\dagger\}exp\{tX\} = \mathbb{1} + t(X^\dagger + X) + O(|t^2|),$$

and hence we see that $X^\dagger + X$ must vanish, thus deducing that the matrices of the Lie algebra $\mathfrak{su}(2)$, are precisely those that appart from being traceless are skew-Hermitian.

All in all, we can write all the information that we have obtained in the form of a diagram of groups and algebras, where ι represent the natural inclusions and exp the exponential mapping.

$$\begin{tikzcd} \mathfrak{su}(2) \arrow[r, "\iota"] \arrow[d, "exp"] & \mathfrak{sl}(2,\mathbb{C}) \arrow[d, "exp"] \\ SU(2) \arrow[r, "\iota"] & SL(2,\mathbb{C}) \end{tikzcd}$$

To understand this relationships, appart from the abstract diagram, that helps us get a global picture of the situation, we will write the explicit generators for our algebras, and this will also allow us to make concrete computations, in math and physics.

To do this, we start by noting that the algebra $\mathfrak{sl}(2,\mathbb{C})$ has complex dimension 3. There are various ways to see this, and in fact giving the explicit basis as we will is one of them, but appart from this, one can appeal to a more differential-gemetric perspective on the dimension. And just see that the dimension of $SL(2,\mathbb{C})$ as a complex manifold is 3, this can be seen by using the "preimage theorem", that is just a consequence of the implicit function theorem and also works for complex manifolds.

Thus by considering the determinant on 2×2 complex matrices as a holomorphic function

$$det\colon End(\mathbb{C}^2) \to \mathbb{C},$$

note that our group of interest is just the preimage of a point, meaning $SL(2,\mathbb{C}) = det^{-1}(1)$. In fact the point $1 \in \mathbb{C}$ is a *regular value* for the determinant, meaning that for every point in the preimage $(SL(2,\mathbb{C})$, the differential map is surjective, which just amounts to having some non-zero coordinate, and this happens for every matrix different from the zero matrix. Thus $1 \in \mathbb{C}$ being a regular value implies that our group $SL(2,\mathbb{C})$ is a complex manifold of dimension $dim_{\mathbb{C}}End(\mathbb{C}^2) - dim_{\mathbb{C}}\mathbb{C} = 4 - 1 = 3$ (this dimension is given by the regular value theorem). Thus also we can conclude that its Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ also has $dim_{\mathbb{C}} = 3$.

The basis of this algebra can be given thus by the following three generators

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which are clearly elements of $\mathfrak{sl}(2,\mathbb{C})$ since they are traceless, and furthermore linearly independent over $\mathbb{C}$, thus defining a basis for our algebra. We have the following commutation relations:

$$\begin{aligned}
[X,Y] &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = H, \\
[H,X] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 2X, \\
(H,Y] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} = -2Y.
\end{aligned}$$

Now we want to understand, with this basis, how does $\mathfrak{su}(2)$ sit inside of the algebra $\mathfrak{sl}(2,\mathbb{C})$. First we note that as vector spaces, $\mathfrak{su}(2)$ is just a real-subspace (meaning that as vector spaces $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C})$, and in terms of the previous basis we can give a basis (over $\mathbb{R}$) of $\mathfrak{su}(2)$, concretely:

$$iH = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}; X - Y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; i(X+Y) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

form a basis of $\mathfrak{su}(2)$. These matrices are extremely useful for doing computations, and physicists denote a related basis as the *Pauli matrices*, that are usually denoted by σ_i.

Now, given this basis of the algebra, we note that the generator $i(X+Y)$ generates a certain one-parameter subgroup of $SU(2)$ that is given by

$$exp\{it(X+Y))\} = \begin{bmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{bmatrix},$$

this can be seen either by a direct computation, and the result will just follow by applying Euler's identity or just by noting that both one-parameter subgroups come from the same generator, meaning

$$\frac{d}{dt}\begin{bmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = i(X+Y)$$

.

With this in our hand, we consider the group element

$$w = exp\{\frac{1}{2}i\pi(X+Y)\} = \begin{bmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{bmatrix}\Bigg|_{t=\frac{\pi}{2}} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

and it's associated action by conjugation in the algebra $\mathfrak{sl}(2,\mathbb{C})$, gives rise to what is usually called the *Weyl group reflection*, due to its action on the generators:

$$wHw^{-1} = -H; wXw^{-1} = Y; wYw^{-1} = X,$$

this just means that conjugation by w in our algebra just interchanges the generators in the following way

$$\begin{aligned} H &\to -H \\ X &\to Y \\ Y &\to X. \end{aligned}$$

If we now turn back our attention to the diagram we previously found, namely

$$\begin{tikzcd} \mathfrak{su}(2) \arrow[r, "\iota"] \arrow[d, "exp"] & \mathfrak{sl}(2,\mathbb{C}) \arrow[d, "exp"] \\ SU(2) \arrow[r, "\iota"] & SL(2,\mathbb{C}), \end{tikzcd}$$

one can note that for every object in this diagram, representations on a complex vector space V can be considered, and our goal now is to see if we can relate the various representations of this objects. If we denote by $(\mathfrak{su}(2) \to End(V))_{\mathbb{R}}$ the $\mathbb{R}-$linear algebra homomorphisms, by $(\mathfrak{sl}(2,\mathbb{C}) \to End(V))_{\mathbb{C}}$ the $\mathbb{C}-$linear algebra homomorphisms, by $(SU(2) \to GL(V))_{\mathbb{R}}$ the real-analytic group homomorphisms and by $(SL(2,\mathbb{C}) \to GL(V))_{\mathbb{C}}$ the complex-analytic group homomorphisms, then we can relate them by the diagram

$$\begin{tikzcd}
(\mathfrak{su}(2) \to End(V))_{\mathbb{R}} & (\mathfrak{sl}(2,\mathbb{C}) \to End(V))_{\mathbb{C}} \arrow[l, "r_1"'] \\
(SU(2) \to GL(V))_{\mathbb{R}} \arrow[u, "d"] & (SL(2,\mathbb{C}) \to GL(V))_{\mathbb{C}}, \arrow[l, "r_2"'] \arrow[u, "d"]
\end{tikzcd}$$

where the map d is just the derivative and the maps r_1 and r_2 are the natural restrictions. Recall that the Lie algebra is no more than the tangent space at the identity of the group, and the derivative d of the representation of the group $\rho\colon G \to GL(V)$ induces a representation of the associated algebra in a natural way.

We now come to an important result, that in part justifies our previous need to understand the subgroup $SU(2)$ and its algebra, since it will tell us that the "diagram of representation" above is in fact a correcpondence (it is bijective!).

Lemma 3.5.3. *In the diagram above, all the maps d, r_1 and r_2 are bijective, meaning that there is a one-to-one correspondence between representations of $SL(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C}), SU(2), \mathfrak{su}(2)$.*

The proof can be seen in (Chap. III Proposition 3.1) [1].

With this result in our hands, we have that the representations of our algebra of interest, namely $\mathfrak{sl}(2,\mathbb{C})$ are in fact in a bijective correspondence with representations of $SU(2)$, which is a **compact Lie group**. And in the setting of compact Lie groups, there is a huge machinery of theory developed to understand their representations. For instance, we have a fundamental result for the theory of representations of compact Lie groups due to Hermann Weyl that deals with the problem of characterizing representations, in fact it will tell us that finite-dimensional representations will be completely reducible for our algebra $\mathfrak{g} = SL(2,\mathbb{C})$. We could prove this fact for the particular compact Lie group of interest, namely $SU(2)$, but without a lot of extra effort the general proof for an arbitrary compact Lie group can be given, and thus we will give the general result, that we will later apply to our group.

Theorem 3.5.1. *If G is a compact Lie group and $\rho\colon G \to GL(V)$ is a finite-dimensional representation on a complex vector space V, then the representation ρ is completely reducible.*

Proof. We start by choosing a basis for V so that we can identify $V \cong \mathbb{C}^n$. We denote by dg the natural left-invariant measure on the Lie Group G. This measure can be constructed from left-invariant differential forms on the manifold. Then

$$M(g) = \rho(g)\rho(g)^*,$$

is a positive definite Hermitian matrix for every group element $g \in G$. We define

$$M = \int_G M(g)dg,$$

and we have that M is also a positive definite Hermitian matrix. Now we consider

$$\begin{aligned}\rho(g)M\rho(g)^* &= \int_G \rho(g)\rho(\tau)\rho(\tau)^*\rho(g)^* d\tau \\ &= \int_G \rho(g\tau)\rho(g\tau)^* d\tau = \int_G \rho(\tau)\rho(\tau)d\tau = M,\end{aligned}$$

where we have used the invariance of the measure $d\tau$ under the action of G on itself by left translation.

Due to the fact that M is positive definite, we can write it as

$$M = NN^*$$

for a certain positive definite matrix N. We have that $\tilde{\rho} = N^{-1}\rho N$ is equivalent to ρ, and furthermore we compute

$$\tilde{\rho}(g)\tilde{\rho}(g)^* = (N^{-1}\rho(g)N)(N^{-1}\rho(g)N)^* = N^{-1}\rho(g)NN^*\rho(g)^*(N^{-1})^* = N^{-1}M(N^{-1})^* = Id,$$

and thus we see that $\tilde{\rho}(g)$ is a unitary matrix for every $g \in G$. Now we want to see that $\tilde{\rho}$ is completely reducible. Suppose that $V_0 \subset V$ is an invariant subspace under the action of $\tilde{\rho}$. We denote, as usual, by $V_0^{\perp}$ the orthogonal complement to V with respect to the usual Hermitian metric on $\mathbb{C}^n$ (recall that we started the proof by making the identification). We then have that $\tilde{\rho}(V_0) \subset V_0$, and from this follows, due to the fact that we are dealing with a unitary representation ($\tilde{\rho}$), that $\tilde{\rho}(V_0^{\perp}) \subset V_0^{\perp}$. □

This technique was introduced by H. Weyl, and due to the fact that it reduces an arbitrary representation to an equivalent unitary one, the technique is usually called the *unitary trick*.

From this general theorem, by applying it to the case of our particular compact Lie group of interest, namely $G = SU(2)$, we obtain a very interesting result about finite-dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$.

Corollary. If ρ is a finite-dimensional representation of the algebra $\mathfrak{sl}(2, \mathbb{C})$, then ρ is completely reducible.

Now that we know that every finite-dimensional representation is built as a direct sum of irreducible representations, we would like to characterize this irreducible representations. And it turns out that one can give a complete characterization of this irreducible representations, up to equivalence, that in turn gives us the structure of all possible finite-dimensional representations of our algebra $\mathfrak{sl}(2, \mathbb{C})$. So, we devote ourselves now to characterizing irreducible representations.

Definition 3.5.4. Consider ρ to be a finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ on a complex vector space V. We will specially be interested in the possible eigenvalues/eigenvectors for the operator $\rho(H)$. And thus for $\lambda \in \mathbb{C}$ we define

$$V^{\lambda} = \{v \in V \colon \rho(H)v = \lambda v\}.$$

We call $v \in V^{\lambda}$ an element of *weight* λ. A vector $0 \neq v \in V^{\lambda}$ with $\rho(X)v = 0$ is said to be *primitive of weight* λ.

We will now prove a pair of lemmas that will help us in understanding the possible irreducible representations. As usual, we denote by ρ a finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ on a complex vector space V.

Lemma 3.5.4.
- *$\bigoplus_{\lambda\in\mathbb{C}} V^{\lambda}$ is a direct sum,*
- *If $v \in V^{\lambda}$ (is of weight λ), then $\rho(X) \in V^{\lambda+2}$ (is of weight $\lambda+2$) and $\rho(Y)v \in V^{\lambda-2}$ (is of weight $\lambda-2$).*

Proof. The first fact just follows from the fact that eigenvectors corresponding to different eigenvalues are linearly independent. For the second part, we compute

$$\begin{aligned}\rho(H)\rho(X)v &= (\rho(H)\rho(X) - \rho(X)\rho(H))v + \rho(X)\rho(H)v \\ &= \rho([H,X])v + \lambda\rho(X)v = \rho(2X)v + \lambda\rho(X)v = (\lambda+2)\rho(X)v.\end{aligned}$$

And similarly we deduce that $\rho(H)\rho(Y)v = (\lambda-2)\rho(Y)v$. □

Lemma 3.5.5. *If ρ is a finite-dimensional (complex) representation of $\mathfrak{sl}(2,\mathbb{C})$ then it has at least one primitive vector.*

Proof. We start by considering v_0 an eigenvector of $\rho(H)$, and we consider the sequence of eigenvectors of $\rho(H)$ given by

$$v_0, \rho(X)v_0, \rho(X)^2 v_0, ..., \rho(X)^n v_0, ...$$

Now, clearly the nonzero terms in this sequence are linearly independent (due to the previous lemma). Hence the sequence must terminate, meaning that for some k, $\rho(X)^k v_0 = 0, \rho(X)^{k-1}v_0 \neq 0$. This just means that we have constructed a primitive vector $v = \rho(X)^{k-1}v_0$.

□

We are now ready to see some of the basic information about irreducible representations of the algebra $\mathfrak{sl}(2,\mathbb{C})$. We present this information in the form of a theorem.

Theorem 3.5.2. *Let ρ be an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ on a finite-dimensional complex vector space V. Let $v_0 \in V$ be a primitive vector of weight λ for the representation ρ. Then, by letting $v_{-1} = 0$, and setting*

$$v_n = \frac{1}{n!}\rho(Y^n)v_0,$$

and we obtain

1. $\rho(H)v_n = (\lambda - 2n)v_n$,
2. $\rho(Y)v_n = (n+1)v_{n+1}$,
3. $\rho(X)v_n = (\lambda - n + 1)v_{n-1}$.

Moreover, $\lambda = m$, where $m + 1 = dim_{\mathbb{C}}V$, and

$$\rho(Y^n)v_0 = 0; \quad n > m.$$

The proof of this theorem consists in just putting together the results above and using the finite-dimensionality of V and irreducibility. We omit the proof, the interested reader is referred to (Chap. III Theorem 3.7) [1].

From this theorem one can also deduce that there is, up to equivalence, only one irreducible representation of dimension $m + 1$.

And now we want to consider a specific representation of $\mathfrak{sl}(2,\mathbb{C})$ on the exterior algebra of forms on a Hermitian vector space E. We associate to E the algebra of forms $\wedge F$ and the operators L and L^*. We introduce the following notation

$$\Lambda \colon = L^*$$

$$B \colon = \sum_{p=0}^{2n}(n-p)\Pi_p.$$

Equipped with this notation, we define the representation

$$\alpha \colon \mathfrak{sl}(2,\mathbb{C}) \to End(\wedge F)$$

by

$$\alpha(X) = \Lambda; \;\; \alpha(Y) = L; \;\; \alpha(H) = B.$$

One can check that the appropriate commutation relations are satisfied, and thus α is indeed a representation.

Definition 3.5.5. A $p-$form $\varphi \in \wedge^p F$ is said to be *primitive* if $\Lambda\varphi = 0$, meaning $\alpha(X)\varphi = 0$.

Note that all the operators involved are real operators, and thus the action of α leaves the real forms $\wedge_{\mathbb{R}}F$ invariant. The decomposition of this exterior algebra into irreducible components is the so-called *Lefschetz decomposition.* This decomposition is compatible with the bidegree decomposition due to the fact that L, Λ and B are bihomogeneous operators. From now on we will use the notation $(x)^+ = max(x, 0)$.

Now, by using the facts we know about representations of the algebra $\mathfrak{sl}(2, \mathbb{C})$ we get the following theorem.

Theorem 3.5.3. *If E is a Hermitian vector space of complex dimension n, then the following holds:*

1. *If $\varphi \in \wedge^p F$ is a primitive $p-$form then $L^q = 0$ for $q \geq (n-p+1)^+$.ç*
2. *There are no primitive forms of degree $p > n$.*

The next theorem is called the *Lefschetz decomposition theorem* for an Hermitian exterior algebra.

Theorem 3.5.4. *Let E be an Hermitian vector space of complex dimension n, and consider a $p-$form $\varphi \in \wedge^p F$ then*

1. *φ can be written uniquely in the form*

$$\varphi = \sum_{r \geq (p-n)^+} L^r \varphi_r,$$

   where φ_r is a primitive $(p-2r)-$form.

2. *If $L^m \varphi = 0$, then the primitive forms φ_r vanish if $r \geq (p-n+m)^+$.*
3. *If $p \leq n$, and $L^{n-p}\varphi = 0$, then $\varphi = 0$.*

This results come from the previous facts about the representations of the algebra, for more details on the proof the interested reader is referred to [1] (Chap. V Theorem 3.2).

This algebraic result will be of use in the next section, where we will use it to see the *Lefschetz decomposition* in the geometric setting.

3.5.2 Kähler manifolds

Now we turn to a fundamental object of study in the theory of complex manifolds. They are a special type of manifolds that appear in a lot of geometric situations and that specially in the compact case we can analyze them deeply, due to all the structure they will turn out to have. They are a natural arena for one to see the interplay between Riemannian geometry, symplectic geometry and complex geometry, where the three geometries overlap. This is due to heavy theoretical reasons but put simply, one of the previous two geometries determines the third, and thus all the three fields of geometry come together in the study of *Kähler manifolds*. This should be a reason in itself to study them, but apart from this they will turn out to have much more special properties, and we will have a good place to understand what the theory of differential operators can tell us about the geometry.

Let X be a Hermitian complex manifold with hermitian metric h. Associated to (X, h) we have a *fundamental form* Ω, that at each point $p \in X$ is just the form of type $(1,1)$ which is the fundamental form (we defined before) associated with the Hermitian form

$$h_x \colon T_xX \times T_xX \to \mathbb{C},$$

that is given by the metric h.

Definition 3.5.6. We call the metric h on X a *Kähler metric* if the fundamental form Ω associated to h is closed:

$$d\Omega = 0.$$

We call a complex manifold X that admits at least one complex structure a manifold of *Kähler type*. And a complex manifold equipped with a Kähler metric is called a *Kähler manifold*.

This definition restricts the kind of manifolds we study since **not every complex manifold admits a Kahler metric**.

We now want to see how these object look like locally, the metric h and the associated form Ω. If one considers complex coordinates z_j, then the metric h can be written as

$$h = \sum_{\mu\nu} h_{\mu\nu}(z) dz_\mu \otimes d\overline{z}_\nu,$$

where $(h_{\mu\nu}(z))$ is a positive-definite Hermitian matrix for every z and the fundamental form Ω is written in this coordinates as

$$\Omega = \frac{i}{2} \sum_{\mu\nu} h_{\mu\nu}(z) dz_\mu \wedge d\overline{z}_\nu.$$

In fact it should be noted that one can compute the coefficients just by

$$h_{\mu\nu}(z) = h(\frac{\partial}{\partial z_\mu}, \frac{\partial}{\partial \overline{z}_\nu})$$

Now we want to see some examples of Kähler manifolds that will show us that they occur naturally in a variety of geometric situations.

Example 3.2. We consider $X = \mathbb{C}^n$, endowed with the usual coordinates $z_\mu = x_\mu + iy_\mu$ and with the metric $h = \sum_{\mu=1}^n dz_\mu \otimes dz_\mu$. In this setting the fundamental form Ω is just

$$\Omega = \frac{i}{2} \sum_{\mu=1}^{n} dz_\mu \wedge dz_\mu = \sum_{\mu=1}^{n} dx_\mu \wedge dy_\mu,$$

this form is just the canonical symplectic form for X viewed as the real manifold $\mathbb{R}^{2n}$. And clearly this form is closed since Ω is constant. And hence $\mathbb{C}^n$ is a Kähler manifold.

Other common examples are for instance Elliptic curves and Projective spaces.

Now that we have in our hands some important examples of Kähler manifolds, we turn to a proposition that allows us to get, from this examples that we already have, a wide variety of examples, just by considering submanifolds.

Lemma 3.5.6. *If X is a Kähler manifold with Kähler metric h and $M \subset X$ is a complex submanifold, then h induces a Kähler metric on M.*

Proof. Since M is a complex submanifold of X, we can consider the injection map

$$\iota \colon M \to X.$$

Then we just get the desired structure on M by pullback along this injection. This just means that $h_M = \iota^* h$ induces a hermitian metric on M and $j^*\Omega = \Omega_M$ is just the fundamental form associated with the pullback metric h_M. Now it remains to check that this really defines a Kähler metric, meaning that

$$d\Omega_M = 0.$$

But this is just a consequence of the fact that the exterior derivative commutes with the pullback, giving us

$$d\Omega_M = d\iota^*\Omega = \iota^* d\Omega = 0,$$

since we asumed the original form Ω to be closed. □

Hence now we can conclude that any complex submanifold of either $\mathbb{C}^n$, $\mathbb{CP}^n$ or complex tori are Kähler manifolds. And hence a wide variety of examples pop out, just by considering submanifolds of those manifolds.

Recall that on the context of a complex manifold, we have the usual exterior derivative d as well as the Dolbeaut operators ∂ and $\overline{\partial}$. From this operators we can construct the associated laplacians, that we denote by

$$\begin{aligned}
\Delta &= dd^* + d^*d \\
\square &= \partial\partial^* + \partial^*\partial \\
\overline{\square} &= \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}
\end{aligned}$$

we know from ordinary de Rahm cohomology and Hodge theory that the operator d and its associated laplacian play an important role in differential geometry, and in the study of the topology of smooth manifolds.

In our case, the laplacian operator $\overline{\Box}$, associated to the operator $\overline{\partial}$ will also play an important role in the study of Kähler manifolds. We just note that the notation is not artificial and that the operator $\overline{\Box}$ is in fact the complex conjugate of the operator $\Box$. And now we come to an important question, what is the relationship between this laplacians? In a general complex manifold there is a complicated relationship between them involving the torsion tensor, hence we can not say a lot about them since the torsion tensor in itself can incorporate a lot of information. But in the case of a Kähler manifold, it turns out that we have a very concrete and clear relationship. We recall that an operator $A\colon \mathcal{E}^*(X) \to \mathcal{E}^*(X)$ is said to be *real* if $\overline{A\omega} = A\overline{\omega}$, meaning $\overline{A} = A$. We turn now to the fundamental result characterizing the three laplacians in the case of a Kähler manifold.

Theorem 3.5.5. *Let X be a Kähler manifold with Kähler metric h. Then if we define the differential operators $d, d^*, \overline{\partial}, \overline{\partial}^*, \partial, \partial^*, \Delta, \Box, \overline{\Box}$, with respect to the metric h, we get that Δ commutes with $*, d$ and L, furthermore we have the identity*

$$\Delta = 2\Box = 2\overline{\Box}.$$

In particular this means that

- *Both laplacians $\overline{\Box}$ and $\Box$ are real operators.*
- $\Delta\big|_{\mathcal{E}^{p,q}} \colon \mathcal{E}^{p,q} \to \mathcal{E}^{p,q}$.

Remark. Neither of the last properties are satisfied generically for complex manifolds, and this will imply certain *topological restrictions* on Kähler manifolds, as we will see later.

We also note that the theorem does not assume compactness of the manifold X. We have mainly work under the compact setting thus not having to worry about the analytic details that non-compact manifolds have. But the result is the same provided that in the non-compact case one restricts to forms with compact support, and the formal adjoints are defined in the same way.

Due to the importance of this theorem, our clear goal now it to prove it. To accomplish this, it turns out that we will need some of the representation theory that we developed previously for the lie algebra $\mathfrak{sl}(2, \mathbb{C})$, applying it to the study of the various differential operators that appear in the theorem. We will use the operators L and L^* as auxiliary tools, and since they allow us to introduce into the complex manifold-setting the notion of a *primitive differential form.*

Definition 3.5.7. Given the operator L, and it's associated adjoint $L^* = w * L*$ on a certain complex manifold X. We say that a differential form $\alpha \in \mathcal{E}^p(X)$ is a *primitive differential form of degree p* if

$$L^*\alpha = 0.$$

And we shall denote by $\mathcal{E}_0^p(X)$ the vector space of primitive p-forms. This shall not be confused with differential forms with compact support, that usually also have the same notation.

Now what we have done it to bring into the (complex)differential-geometric setting all the algebraic notions that we had in the previous section, and thus all the results that apply to primitive forms on a Hermitian vector space then **apply to primitive differential forms**. Thus the algebraic results that were seen previously in the algebraic setting also hold (similarly) in this geometric arena.

Before we start we need to define some additional operators which are real operators that are very useful in applications that involve integration and the Stokes' theorem.

Definition 3.5.8. We define the operators d_c and d_c^* by

$$\begin{aligned} d_c &= J^{-1} dJ = wJdJ \\ d_c^* &= J^{-1} d^* J = wJd^* J. \end{aligned}$$

Remark. It is important to see the alternative expression that the operator d_c has, namely for a given differential form α we have

$$\begin{aligned} d_c\alpha = wJdJ\alpha &= (-1)J(\partial\alpha + \overline{\partial}\alpha) \\ &= (-1)(i\partial\alpha - i\overline{\partial}\alpha) = -i(\partial - \overline{\partial})\alpha, \end{aligned}$$

and one could have used this last expression as the definition (for the case of a complex manifold). From this expression we derive

$$dd_c = (\partial + \overline{\partial}) - i(\partial - \overline{\partial}) = -i(\partial^2 + \overline{\partial}^2 + \overline{\partial}\partial - \partial\overline{\partial}) = 2i(\partial\overline{\partial}),$$

which is just a $(1,1)$ operator acting on the differential forms on our manifold $\mathcal{E}^*(X)$.

The natural thing to study when one has a variety of different (differential) operators as we have now is their commutation relations, we want to see what operators commute and which don't, and it will be enlightening to know what this commutation relations are as this will ease our proof of the theorem. So we are now going to devote some time to understanding the commutation relations that exist between the various operators we defined (specially in the case of Kähler manifolds). We start with an important result.

Lemma 3.5.7. *If X is a Kähler manifold then the following commutation relations hold:*

- $[L, d] = 0$ *and* $[L^*, d^*] = 0$.
- $[L, d^*] = d_c$ *and* $[L^*, d] = -d_c$.

The proof of this can be seen in (Chap. V Theorem 4.8) in [1].

From this result we can extract some corollaries, that will tell us about the commutation relations for other operators of interest.

Corollary. If X is a Kähler manifold, then the following commutation relations hold:

$$[L, d_c] = 0, [L^*, d_c^*] = 0, [L, d_c^*] = -d, [L^*, d_c] = d^*.$$

This just follows directly from the previous result.

And now by using the fact that in a complex manifold we have a bidegree on our differential forms, we obtain the following.

Corollary. If X is a Kähler manifold, then the folllowing commutation relations hold:

$$[L, \partial] = [L, \overline{\partial}] = [L^*, \partial^*] = [L^*, \overline{\partial}^*] = 0$$

$$[L, \partial^*] = i\overline{\partial}, [L, \overline{\partial}^*] = -i\partial$$

$$[L^*, \partial] = i\overline{\partial}^*, [L^*, \overline{\partial}] = -i\partial^*$$

$$d^* d_c = -d_c d^* = d^* L d^* = -d_c L^* d_c$$

$$dd_c^* = -d_c^* d = d_c^* L d_c^* = -dL^* d$$

$$\partial\overline{\partial}^* = -\overline{\partial}^* \partial = -i\overline{\partial}^* L\overline{\partial}^* = -i\partial L^* \partial$$

$$\overline{\partial}\partial^* = -\partial^* \overline{\partial} = i\partial^* L\partial^* = i\overline{\partial} L^* \overline{\partial}.$$

To show this one just uses the commutation relations from the previous results and compares bidegrees.

Although at first sight it would seem that we have just seen a bunch of commutation relations between the differential operators of our manifold, and that we are still far from proving the theorem we wanted to prove, actually this commutation relations are a big ammount of information, and will allow us in fact to prove the result just now.

Proof. (Theorem 3.5.5):

From the definition of d^* and Δ we have that Δ commutes both with d^* and $*$. We have to see that it also commutes with L. To do this we just compute and use the commutation relations from above:

$$\begin{aligned} \Delta L - L\Delta &= dd^* L + d^* dL - Ldd^* - Ld^* d \\ &= dd^* L + d^* Ld - dLd^* - Ld^* d \\ &= -d[L, d^*] - [L, d^*]d, \end{aligned}$$

and using the relations above we deduce

$$\Delta L - L\Delta = -dd_c - d_c d.$$

Now, since d and d_c anticommute, due to the relationship $\partial\overline{\partial} + \overline{\partial}\partial = 0$, we get

$$\Delta L - L\Delta = 0.$$

Now what remains to be seen is the relationship between the various laplacians. First, we expand Δ in terms of operators that can be used later for comparison:

$$\Delta = dd^* + d^*d = d[L^*, d_c] + [L^*, d_c]d$$

Note here that the information about the metric in the laplacian Δ is contained in L^*, due to the fact that d, d_c only depend on the differentiable and complex structures. We multiply on the left by J^{-1} and on the right by J to obtain

$$\Delta_c = -d_c L^* d + d_c d L^* - L^* d d_c + d L^* d_c.$$

Since d and d_c anticommute we obtain that $\Delta_c = \Delta$.

We now expand the following expression (noting that $2\partial = d + id_c$):

$$\begin{aligned} 4(\partial\partial^* + \partial^*\partial) &= (d + id_c)(d^* - id_c^*) + (d^* - id_c^*)(d + id_c) \\ &= (dd^* + d^*d) + (d_c d_c^* + d_c^* d_c) + i(d_c d^* + d^* d_c) - i(dd_c^* + d_c^* d). \end{aligned}$$

Now by the previous corollaries, the two last expressions vanish. We also have

$$\Delta_c = J^{-1}\Delta J = J^{-1}dd^*J + J^{-1}d^*dJ = d_c d_c^* + d_c^* d_c.$$

And therefore we have that

$$4\Box = \Delta + \Delta_c + 0 = 2\Delta.$$

And hence we have seen that $2\Box = \Delta$. The equality for the $\overline{\Box}$ laplacian is proven in a similar manner and we omit that.

Now, note that since $\Box$ is an operator of bidegree $(0,0)$ we deduce that Δ is also an operator of bidegree $(0,0)$. And due to the fact that Δ is a real operator, one deduces that both $\Box$ and $\overline{\Box}$ are also real operators.

$\Box$

Now that we have the theorem in our hands, and have seen the importance that the commutation relations have, we can deduce from the theorem more information, and you guessed it, even more commutation relations! But the ones that we will see now are special, since we will see that the usual laplacian Δ, which is used to define the harmonic theory in our manifold, giving us also the topology of the manifold, commutes with a lot of operators. And although this may not seem like a big thing, due to the big mountain of commutation relations that we have been building, it will turn out to have a lot of important implications at the level of cohomology.

Corollary. If X is a Kähler manifold, then the usual laplacian Δ commutes with the operators $J, L^*, d, \partial, \overline{\partial}, \partial^*$ and d^*.

Now, although as we said this corollary seems not to contain a huge amount of new information about anything, it turns out that if one carefully looks at the result, one can extract a lot of information.

First we note that, since we have seen that the laplacian operator Δ commutes with L^*, we can go back to the previous section and deduce that we will have an analogue of the *Lefschetz decomposition* that we found for Hermitian exterior algebras in this geometric setting.

By Theorem 3.5.5, in a Kähler manifold, the harmonic differential forms with respect to Δ are just the same as the $\Box-$harmonic forms and the $\overline{\Box}-$ harmonic forms. Thus, in a Kähler manifold, we can refer to the *harmonic forms* on X, without specifying with respect to what operator since they all turn out to be the same, due to the theorem. These harmonic forms will be denoted by $\mathcal{H}^l(X)$ and $\mathcal{H}^{p,q}(X)$. We can also impose the *primitive* condition on this forms, meaning that we will want to consider the spaces of *primitive harmonic forms*, that we denote by $\mathcal{H}_0^l(X)$ and $\mathcal{H}_0^{p,q}(X)$. They can naturally just be realized as kernels, since by definition they are the forms anihilated by L^*, thus obtaining

$$\begin{aligned}\mathcal{H}_0^l(X) &= \ker(L^*\colon \mathcal{H}^l(X) \to \mathcal{H}^{l-2}(X))\\ \mathcal{H}_0^{p,q}(X) &= \ker(L^*\colon \mathcal{H}^{p,q}(X) \to \mathcal{H}^{p-1,q-1}(X)).\end{aligned}$$

Now we can just use the primitive *Lefschetz decomoposition* of the previous section, together with the fact that, as we have seen, the laplacian operator Δ commutes both with L and L^* in a Kähler manifold, thus inducing the following decomposition in cohomology.

Theorem 3.5.6 (The Lefschetz decomposition theorem)**.** *If X is a compact Kähler manifold of complex dimension n, then we have the following decompositions into direct sums for cohomology.*

$$\begin{aligned}\mathcal{H}_0^l(X) &= \bigoplus_{j\geq (l-n)^+} L^j\mathcal{H}_0^{l-2j}\\ \mathcal{H}_0^{p,q}(X) &= \bigoplus_{j\geq (p+q-n)^+} L^j\mathcal{H}_0^{p-j,q-j}.\end{aligned}$$

And moreover, one can give a more explicit description of this maps via the following result (which is also due to Lefschetz).

Corollary. If X is a compact Kähler manifold of complex dimension n, with metric h and fundamental form Ω, then we have that the map

$$L^{n-p} = e(\Omega^{n-p})\colon H^p(x,\mathbb{C}) \to H^{2n-p}(X,\mathbb{C})$$

is an isomorphism.

Remark. This can be seen just as a version of the Poincaré duality in this setting, and is usually referred to, in algebraic geometry, as the "strong Lefschetz theorem", that Groethendick called in french *Théorème de Lefschetz vache.*

3.6 Hodge-theoretic decomposition on a Compact Kähler manifold

In this section we will state and provide a proof of the Hodge decomposition theorem, in the geometric setting of Kähler manifolds. There are analogous results for the real setting, in fact it is the original setting of Hodge theory. The fact that we are in a Kähler manifold will "amplify" in a sense the decomposition theorem. This work was done mainly by the Japanese mathematician Kunihiko Kodaira.

Recall that if X is a compact complex manifold, we then have the de Rham cohomology groups $H^l(X, \mathbb{C})$, represented by $d-$closed differential forms (with complex coefficients). And we also have the Dolbeaut groups $H^{p,q}(X)$, represented by $\overline{\partial}-$closed $(p,q)-$forms. In a general complex manifold, if α is a $\overline{\partial}-$closed $(p,q)-$form on X, then α need not be $d-$closed and conversely, if β is a $d-$closed k-form on our space X, with the corresponding splitting

$$\beta = \beta^{k,0} + \beta^{k-1,1} + ... + \beta^{0,k}$$

then the components $\beta^{p,q}$ need not be $\overline{\partial}$ closed. And hence there is a non-trivial relationship between both notions of closedness of differential forms. The remarkable fact is that in the case of Kähler manifolds, we have a much better understanding of their relationship, given in the following theorem.

Theorem 3.6.1 (Hodge decomposition)**.** *If X is a compact Kähler manifold, then we have a direct sum decomposition*

$$H^l(X, \mathbb{C}) \cong \bigoplus_{p+q=l} H^{p,q}(X),$$

moreover,

$$\overline{H}^{p,q} = H^{q,p}.$$

We note that the isomorphism giving the decomposition in the theorem is actually an equality if one considers the cohomology via their harmonic representatives, as we shall do in the proof.

Proof. What we will show is that the direct decomposition holds for the harmonic representatives, meaning that

$$\mathcal{H}^l = \bigoplus_{p+q=l} H^{p,q}(X),$$

and thus the result will follow due to the isomorphism between cohomology and harmonic forms.

For proving this equality we start with a form $\alpha \in \mathcal{H}^l(X)$. This means that it is harmonic, meaning $\Delta\alpha = 0$. But we know from the previous section that in a Kähler manifold, $2\overline{\Box} = \Delta$. Thus we can conclude that $\overline{\Box}\alpha = 0$. Now, we have that our form α can be writen as a sum

$$\alpha = \alpha^{l,0} + ... + \alpha^{0,l},$$

thus by applying the $\overline{\Box}$ operator, we obtain

$$\overline{\Box}\alpha = \overline{\Box}\alpha^{l,0} + ... + \overline{\Box}\alpha^{0,l}.$$

We saw in the previous section that in a Kähler manifold, the various laplacians preserve the bidegree, and thus from the fact that $\overline{\Box}\alpha = 0$, we deduce that every component is zero, meaning

$$\begin{aligned} \overline{\Box}\alpha^{l,0} &= 0 \\ &\vdots \\ \overline{\Box}\alpha^{0,l} &= 0. \end{aligned}$$

This amounts to saying that we can build a map from $\mathcal{H}^l(X)$ to $\mathcal{H}^{p,q}(X)$ just by the component decomposition, meaning

$$\begin{aligned} \Phi\colon \mathcal{H}^l(X) &\to \bigoplus_{p+q=l} \mathcal{H}^{p,q}(X) \\ \alpha &\to (\alpha^{l,0}, ..., \alpha^{0,l}). \end{aligned}$$

This map is well defined since each of the components is $\overline{\Box}-$harmonic, and thus the whole vector as defined belongs to the direct sum. Now we want to see that this map, as defined, induces the desired isomorphism (and in fact equality if one looks at harmonic $\mathcal{H}^{p,q}$ forms as being inside of $\mathcal{H}^{p+q}$ in the natural way). Since the decomposition of harmonic forms into it's homogeneous parts is a direct sum, the map is clearly injective, since if all the components were zero, the original form would have to be the zero form. Thus we have injectivity, it remains to check that Φ is surjective.

But surjectivity is also really easy to check. If we have a form $\beta \in \mathcal{H}^{p,q}(X)$, meaning $\overline{\Box}\beta = 0$, due to the identities that we know for the laplacians this just means

$$0 = \overline{\Box}\beta = \frac{1}{2}\Delta,$$

and thus one deduces that the form β is also $\Delta-$harmonic, thus meaning that $\beta \in \mathcal{H}^{p+q}(X)$. This just means that we have also seen that Φ is surjective, thus providing the desired isomorphism.

The related fact that $\overline{H}^{p,q}(X) = H^{q,p}(X)$ just comes from the fact that the laplacian $\overline{\Box}$ is a real operator and that complex conjugation gives us an isomorphism from $\mathcal{E}^{p,q}(X)$

into $\mathcal{E}^{q,p}(X)$, this two facts together helps us conclude that conjugation descends to an isomorphism between $\overline{\mathcal{H}}^{q,p}(X)$ and $\mathcal{H}^{p,q}(X)$ due to the fact that the laplacian is a real operator. Thus we have finished the proof.

□

It is important to note that pretty much everything in this proof comes from the theorem on the equivalence of laplacians (and the fact that they are real operators) for X being a Kähler manifold. There is no more to this proof, in a sense it is just a corollary of Theorem 3.5.5.

Now that we have derived this decomposition for Kähler manifolds, we can extract from them some topological consequeces. This will amount to the fact that in a Kähler manifold certain relations between the hodge numbers $h^{p,q}$ and the betti numbers b_l will hold, thus showing us that there are topological restrictions for Kähler structures. More concretely recall what we mean by hodge and betti numbers:

$$b_l(X) = dim_{\mathbb{C}} H^l(X, \mathbb{C}); h^{p,q}(X) = dim_{\mathbb{C}} H^{p,q}(X),$$

we put the various relationships together in a corollary.

Corollary. If X is a compact Kähler manifold, then we have the following relations between beti numbers and hodge numbers:

1. $b_l(X) = \sum_{p+q=l} h^{p,q}(X)$.
2. $h^{p,q}(X) = h^{q,p}(X)$.
3. $b_l(X)$ is even whenever l is odd.
4. $h^{1,0}(X) = \frac{1}{2} b_1(X)$. It is thus a topological invariant!

Proof. The first relationship is deduces just by looking at the dimension in the isomorphism given by the theorem, namely

$$H^l(X, \mathbb{C}) \cong \bigoplus_{p+q=l} H^{p,q}(X).$$

The second relationship is just given by looking at the dimension in the isomorphism induced by complex conjugation, namely

$$\overline{H}^{p,q} = H^{q,p}.$$

The third relationship comes from accurately using the previous two relationships. First we use the first relationship to obtain

$$b_l(X) = \sum_{p+q=l} h^{p,q}(X),$$

now, we concentrate on the case that l is odd, and thus all subspaces come from summing for (p,q) satisfying $p+q=l$. But note that for odd l then when we sum p and q are never equal. And thus by using the second relationship we can deduce that

$$b_l(X) = \sum_{p+q=l} h^{p,q}(X) = \sum_{\substack{p+q=l \\ p<q}} h^{p,q} + h^{q,p} = 2 \sum_{\substack{p+q=l \\ p<q}} h^{p,q}(X),$$

and thus we see that b_l must be even for odd l.

The fourth relationship just comes from looking at the dimensions in the following isomorphism

$$H^1(X,\mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X).$$

□

Thus all of this relationships between the hodge-numbers (comming from the Kähler structure) and the betti numbers (topological) come just as easy dimensional conclusions deduced from the isomorphisms of Theorem 3.6.1. And hence we have seen with this that there are topological restrictions for X to be a compact Kähler manifold, and in fact furthermore, if the manifold X can be equipped with a Kähler structure, then the associated complex structure has to satisfy certain requirements depending on the topology of our manifold.

Now, one should note that there is a very special (and classical) case of Kähler manifold that is of much importance to a lot of areas of mathematics and physics, namely surfaces, *Riemann surfaces*. In fact what we will see now is that one-dimensional complex manifolds will automatically be of Kähler type, and thus this general theory that we built can be applied to surfaces. This are of importance to physics and mathematics in several areas, complex analysis, number theory, string theory...

Theorem 3.6.2. *If X is a complex manifold of complex dimension 1 then it is of Kähler type. And in fact it will be Kähler for every Hermitian metric that we can find in X.*

Proof. Consider on our manifold X an arbitrary Hermitian metric h (just a Hermitian metric for the complex tangent bundle). Then if we see that this metric induces a Kähler structure we will be done. This is clear since if we consider the associated fundamental form Ω, this form is of type $(1,1)$, meaning that it has global degree 2 on our manifold X. Due to the fact that X has only two real dimensions, there are no $3-forms$ on our manifold except for 0, and thus if we take the exterior derivative of the fundamental form, which has to be a $3-form$, it will be zero. Hence we have seen that $d\Omega = 0$ for **for any Hermitian metric** on X. Thus every metric (Hermitian) on a one-dimensional complex manifold is Kähler! And thus our original manifold X is of Kähler type. □

If X is a compact Riemann surface, then it's topology is very restricted, since we have trivial cohomology for degree 3 onwards, and thus we have, using the Hodge decomposition for Kähler manifolds, that

$$H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X).$$

Furthermore we have that $h^{1,0}(X) = h^{0,1}$ (due to the corollary) and thus

$$2h^{1,0}(X) = b_1(X),$$

this means that the hodge number $h^{1,0}(X)$ is in fact a topological invariant for X. In fact this invariant is well known for the study of Riemann surfaces, and it is usually called the *genus*, and is usually denoted by g in the theory of Riemann surfaces. Topologically, the genus classifies all possible compact Riemann surfaces up to isomorphism.

3.7 Application: Embedding Theorems

In this section we will devote to a very concrete application where we will see the power and usefulness of many of the machinery that we have introduced during the course of the sections. The result that we will see is usually called *Kodaira's Embedding Theorem*, and it characterizes compact complex manifolds that admit an embedding into complex projective space. There is a theorem by Chow [48] that we will not prove which asserts that compact complex submanifolds of projective space are actually *algebraic*, meaning that they are defined by the zeroes of homogeneous polynomials. We will accept this theorem, and thus if we see that a certain complex manifold admits an embedding into projective space, we will use Chow's theorem to conclude that it is actually algebraic. Since we are not including in this work techniques from projective algebraic geometry we will not delve deeper into this relationship, but there is a very deep relationship between the field of complex geometry and algebraic geometry. This was emphasized by many mathematicians during the XX'th century, and even before in the XIX'th century, with the work of Riemann for instance. The relationship was further researched by Serre and other french mathematicians, in the *Géometrie Algébrique et Géométrie Analytique* program [50].

But before we start with the problem of embedding complex manifolds into projective space, it is enlightening to see how to deal with the problem of embedding complex manifolds into $\mathbb{C}^N$. Since the main ideas behind both of the problems can be understood in paralel, and one could say that the problem of embedding manifolds into $\mathbb{C}^N$ (what are called *Stein manifolds*) is in a sense "easier" than it's projective counterpart. We will devote now some time to the study of this *Stein manifolds*, by giving some properties and theorems about them that we will not prove but that are useful for seeing the logic behind dealing with embedding problems. Sheafs will turn out to play a very important role, as we will see, and the study of *Stein manifolds* via sheaf-theoretic methods encompasses many classical problems in complex analysis in several variables, and turns out to have some nice applications to solving some classical problems, so we will review the logic behind this ideas, which are mainly due to Henri Cartan.

3.7.1 The case of Stein manifolds

In this section we will briefly see how the theory of sheaves allowed mathematicians in the 50's to show remarkable properties of submanifolds of $\mathbb{C}^N$. Recall that in the real case every manifold can be embedded on euclidean space, but as we said at the start of this work, this is not the case for complex manifolds. Since for instance no (nontrivial) compact manifold can be embedded in euclidean (complex) space due to the maximum principle. Thus the manifolds that we will see in this section will be non-compact. The main reference for this chapter will be [63], we will just comment on results and ideas since we deal with Stein spaces just to see how the machinery of sheaf theory can be of use for dealing with certain problems. The interested reader is referred to [63] for the proofs or more references about this topic.

In 1951, Karl Stein introduced the following class of complex manifolds.

Definition 3.7.1. A complex manifold X is said to be a *Stein manifold* (or holomorphically complete) if the following properties hold:

1. For every pair of points $x \neq y$ in X, there exists a holomorphic function $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$.

2. For every point $p \in X$ there are functions $f_1, ..., f_n \in \mathcal{O}(X)$, $n = dimX$, whose differentials df_j are $\mathbb{C}-$linearly independent at the point p.

3. X is *holomorphically convex*, this is a notion comming from complex analysis in several complex variables. It basically means that for every compact $K \subset X$, the so-called *holomorphically convex hull*

$$\bar{K} = \left\{ z \in X \,\middle|\, |f(z)| \leq \sup_{w \in K} |f(w)| \; \forall f \in \mathcal{O}(X) \right\},$$

is also a compact subset of X.

Note that the second property means that global holomorphic functions provide local charts at every point in the manifold! One can extract some facts from this definition:

- An open set $U \subset \mathbb{C}^n$ is Stein if and only if it is a domain of holomorphy. The notion of holomorphy domain is very important in the theory of several complex variables, and thus Stein manifolds are an intrinsic generalization of this. This equivalence follows from the Cartan/Thullen theorem, which tells us that U is a domain of holomorphy if and only if it is holomorphically convex.

- A Stein manifold does **not** contain any compact complex submanifolds of positive dimension. This is just the same proof we did for showing that there are no complex submanifolds in $\mathbb{C}^n$, it follows from the maximum principle of holomorphic functions.

- The cartesian product of two Stein manifolds is Stein.
- Every closed complex submanifold of a Stein manifold is Stein. In particular, a closed submanifold X of $\mathbb{C}^n$ is Stein (by using coordinate functions restricted to X).
- If $E \to X$ is a holomorphic vector bundle over a Stein space X, then E is also Stein.
- An open Riemann surface is a Stein manifold (nontrivial result due to Behnke and Stein).

The most ramarkable results in the theory of Stein manifolds that we want to comment on are two theorems by Henri Cartan, which give a more abstract and general perspective on a lot of more classical problems. Cartan was one of the first to introduce and use with great hability the theory of sheaves that Leray had invented in the context of complex analysis in several variables. These theorems were proved in his seminar in 1951/1952. One can not overstate the importance that this two theorems have in the development of both analytic and algebraic geometry, since was one of the first successful application of sheaves to solve concrete problems. The theorems are called respectively Theorem A and Theorem B.

Theorem 3.7.1. *(Theorem A) For every coherent analytic sheaf $\mathcal{F}$ on a Stein space $(X, \mathcal{O}_X)$ we have that the stalks are generated, as $\mathcal{O}_{X,x}-$modules by the global sections of $\mathcal{F}$.*

Theorem 3.7.2. *(Theorem B) For every coherent analytic sheaf $\mathcal{F}$ on a Stein space $(X, \mathcal{O}_X)$ we have that*

$$H^p(X, \mathcal{F}) = 0,$$

for every $p \geq 1$.

This cohomological interpretation of the theorem was in part due to Serre, who later generalized this result to algebraic geometry.

This theorems can be applied to a wide variety of results. For instance:

Corollary. Let $\mathcal{F} \xrightarrow{\eta} \mathcal{G}$ be an epimorphism of analytic sheaves over a Stein X. Then, if $Ker(\eta)$ is coherent, then the induced map on global sections $\mathcal{F}(X) \to \mathcal{G}(X)$ is surjective.

Due to the fact that $H^1(X, Ker(\eta)) = 0$, due to Theorem B, the corollary just follows from the exact cohomology sequence

$$\mathcal{F}(X) \to \mathcal{G}(X) \to H^1(X, Ker(\eta)) = 0.$$

Now, if we apply the same reasoning above to the exact sequence

$$0 \to \mathcal{J}_A \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}_A \to 0,$$

where A is a closed complex subvariety of X we obtain the so-called Cartan extension theorem, that one can think of as being a kind of "analytic continuation".

Theorem 3.7.3. *(Extension theorem) Every holomorphic function on a closed complex subvariety of a Stein space X extends to a holomorphic function on X.*

Theorem 3.7.4. *(Cartan's division theorem) If $\mathcal{F}$ is a coherent analytic sheaf on a Stein manifold X with $f_1, ..., f_k \in \mathcal{F}(X)$ generating each stalk $\mathcal{F}_x$, then every global section $f \in \mathcal{F}(X)$ is of the form $f = \sum_j g_j f_j$ for some g_j.*

This also follows from Theorem B, in fact it follows from the corollary above, which precisely justifies the surjectivity.

Theorem 3.7.5. *On a Stein manifold X, the Dolbeaut cohomology groups vanish, meaning that*

$$H^{p,q}_{\overline{\partial}}(X) = 0,$$

for $p \geq 0$ and $q \geq 1$.

Proof. We know that the sheaf Ω^p of holomorphic $p-$ forms on X admits a resolution

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,2} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{p,n} \to 0.$$

Now, since the sheaves $\mathcal{E}^{p,q}$ on X are fine, their cohomology vanishes. And thus we have

$$H^{p,q}_{\overline{\partial}}(X) \cong H^q(X, \Omega^p),$$

by Dolbeaut's theorem. Since the sheaf Ω^p is coherent analytic, these groups vanish. □

Now, we have an embedding theorem for Stein manifolds, that we will not prove, but that goes as follows:

Theorem 3.7.6. *A Stein manifold X of dimension n can be embedded into $\mathbb{C}^{2n+1}$ by a biholomorphic proper map.*

For more information about the embedding of Stein manifolds, the interested reader is referred to [63]. Note that with this result in mind, Stein manifolds are the complex manifolds that behave in a similar way (in the sense of embeddings) as real differentiable manifolds, since one can just think about them as being submanifolds of the Euclidean (complex) space.

We have thus seen the power that sheaf-theoretical results such as Theorems A and B by Cartan can have in tackling a lot of problems, in this case in the setting of Stein manifolds. The theorems are not trivial at all to prove, but once one has those abstract results, one can start proving result after result just by using exact sequences. The key step is to reformulate more "classical" problems in the language of sheaves, once this is done, using this deep results, many apparently difficult result follow in no more than a 2 lines of proof. In the next section we will move from the non-compact setting to the compact, and thus clearly we will not be able to embed compact manifolds in $\mathbb{C}^n$, the theory is a bit more tricky.

3.7.2 Kodaira's embedding theorem

In this section we will characterize manifolds that admit an embedding into projective space. This, together with Chow's theorem, as we said before, characterizes the manifolds thar are indeed projective algebraic, thus giving us a way to relate our ideas to algebraic geometry, and apply algebro-geometric ideas to study complex geometry and viceversa. Although we will not get into those mutual relationships, that are expanded by the *GAGA theorems* [50], it is a powerful setting in which to see the power of the machinery that we have developed in the previous chapters applied to a very concrete mathematical problem, that was solved by Kodaira.

The final characterization turns out to be very clear, a compact Kähler manifold will have such an embedding if and only if it's fundamental form is integral, something that is relatively easy to verify in practice, while showing the existence of such an embedding into projective space can be a very difficult task for a given compact Kähler manifold.

By what we said before, it is clear that we have to restrict our attention to a special class of Kähler manifolds. What we will in fact require (knowing in advance Kodaira's theorem) is an *integrality condition*. Which amounts to saying that a certain differential form is *integral*.

Definition 3.7.2. If α is a $d-$closed differential form on a complex manifold X, α is said to be *integral* if when we look at it's cohomology class $[\alpha] \in H^*(X, \mathbb{C})$, this class $[\alpha]$ belongs to the image of the natural map

$$H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{C}).$$

Remark. This is not the only place where integral differential forms appear in mathematics. For instance we saw before that chern classes are also integral. Dirac's monopole quantization condition can also be seen as an integral condition for a certain differential form. Thus there are many important objects (described by differential forms) both in mathematics and in physics which are integral.

With the notion of integral differential forms, we can make the definition of a *Hodge manifold* (the terminology was first used by André Weil), that turns out to be the necessary notion to be able to characterize compact Kähler manifolds admitting an embedding into projective space.

Definition 3.7.3. If X is a Kähler manifold with Kähler metric h and associated fundamental form Ω, we say that Ω is a *Hodge form* and that h is a *Hodge metric* if Ω is an integral differential form. A Kähler manifold admitting a Hodge metric is called a *Hodge manifold.*

We will give now some terminology for holomorphic line bundles, since they turn out to be a key ingredient in the proof we will give of Kodaira's vanishing theorem. If E is a holomorphic vector bundle then we introduce the following notation

$$E^{\mu} = \bigotimes_{j=1}^{\mu} E$$
$$E^{-\mu} = (E^{*})^{\mu},$$

for every positive integer μ. We denote by $E^0 = X \times \mathbb{C}$ the trivial bundle over X.

There is another important line bundle that we will use, namely what is usually called the *canonical line bundle*

Definition 3.7.4. If X is a complex manifold of complex nimension n, we let

$$K_X : = \bigwedge^{n} T^{*}X$$

and this bundle is usually called the *canonical line bundle of* X.

Remark. Note that if one considers holomorphic sections of this bundle, one obtains the holomorphic $n-$forms, meaning that

$$\mathcal{O}_X(K_X) = \mathcal{O}_X(\bigwedge^{n} T^{*}X) = \Omega_X^n.$$

Whenever the manifold X is fixed, we usually denote the canonical line bundle just by K.

Now we are ready to start our path towards a proof of Kodaira's embedding theorem. To be able to handle this big result, we will need a series of steps:

1. First we need *Kodaira's vanishing theorem*, which plays an analogous role to the Theorem B in the theory of Stein manifolds that we talked about in the previous section. The problem is that, as we will see, things are not that easy in this case. The basic difference is that, for a compact complex manifold, the cohomology groups $H^l(X, \mathcal{O}(E))$ for $l \geq 1$ need not vanish for arbitrary vector bundles E, as is the case for the theory of Stein manifolds. This concrete cohomological fact is what makes this embedding problem more technically difficult.

2. Then we will need some technical lemmas about the behaviour of positive line bundles under quadratic transformations.

3. After that some lemmas that simplify the proof of the embedding theorem.

4. Finally we will be able to give a proof of the Kodaira's embedding theorem.

This will be our guideline in the following, and until the end of this section we will follow this steps.

We thus start with the *vanishing theorem*, that was first proven by Kodaira and that is fundamental for our proof of the embedding theorem. This theorem plays an analogous role to Cartan's theorem B in the theory of Stein manifolds, but the result will not be the same in the compact complex setting, in a sense the geometric situation is not as simple as for Stein manifolds mainly due to the topological restriction of compactness. We will thus formulate the vanishing theorem for line bundles.

We start thus with some definitions (about what we understand by some geometric quantities to be *positive*) that will turn out to be essential for our proof.

Definition 3.7.5. If α is a differential form of type $(1,1)$ on a complex manifold X, it is said to be *positive* if, around every point $p \in X$ one has

$$\alpha(z) = i \sum_{\mu,\nu} \alpha_{\mu\nu}(z) dz_\mu \wedge dz_\nu,$$

with $(\alpha_{\mu,\nu})$ being a positive definite Hermitian symmetric matrix for every point z close to p. Notationally, we will write that α is positive as $\alpha > 0$.

Definition 3.7.6. Consider $E \to X$ to be a holomorphic line bundle over a complex manifold X. And consider the first Chern class $c_1(E)$ viewed as an element of the de Rham cohomology $H^2(X,\mathbb{R})$. Then the bundle E is said to be *positive* if there is a (real) closed differential form α of type $(1,1)$ satisfying $\alpha \in c_1(E)$, and such that α is a positive differential form. We say that the bundle E is *negative* if it's dual E^* is positive.

We now want to characterize the properties that this positive bundles may have, since the definition is not easy to deal with in a concrete setting. It will also be useful to have an alternative vision on positive bundles since it will allow us to compute with our geometric objects in an easier way, as we shall see.

Lemma 3.7.1. *If $E \to X$ is a holomorphic line bundle over a complex manifold X, then the following two conditions are equivalent:*

1. *The bundle E is positive.*
2. *There is an Hermitian metric h on E such that $i\Theta_E$ (the curvature of the canonical connection induced by h) is a positive differential form.*

We won't reproduce the proof, the interested reader is referred to [1] (Chap. VI Proposition 2.2).

We now state the so-called *Vanishing Theorem*, which is originally due to Kodaira [54].

Theorem 3.7.7. *Let X be a compact complex manifold.*

1. *If $E \to X$ is a holomorphic line bundle such that $E \otimes K^*$ is a positive line bundle. Then*
$$H^q(X, \mathcal{O}(E)) = 0, \quad q > 0.$$
2. *If $E \to X$ is a negative line bundle. Then*
$$H^q(X, \Omega^p(E)) = 0, \quad p + q < n.$$

To prove the above theorem we need some inequalities that are due to Nakano. We will quote them without proof and afterwards derive from them the vanishing theorem.

Lemma 3.7.2. *If $\xi \in \mathcal{H}^{p,q}(E)$, for $E \to X$ a holomorphic vector bundle. In this setting we have the operators $L\colon \mathcal{E}^p(X,E) \to \mathcal{E}^{p+2}(X,E)$ and L^* (it's adjoint) naturally extended to $E-$valued forms, and the curvature Θ of the connection associated with the metric. Under this circumstances we have*

1. $\frac{i}{2}(\Theta \wedge L^*\xi, \xi) \leq 0.$
2. $\frac{i}{2}(L^*\Theta \wedge \xi, \xi) \leq 0.$

For the interested reader, the proof is in [1] (Chapter VI. Proposition 2.5).

Now we will prove the vanishing theorem.

Proof. (Kodaira's vanishing theorem) Suppose that E is a negative line bundle. Then one can take as a fundamental form for a Kähler metric on X the following:
$$\Omega = -\frac{i}{2}\Theta.$$
(note that here Θ is a closed form of type $(1,1)$, whose coefficient matric is negative-definite). In the previous lemma (Nakano inequalities), we substract 2. from 1. and thus obtain (note that $-\frac{i}{2}$ is replaced by L)
$$([L^*L - LL^*]\xi, \xi) \leq 0.$$

Now from the commutation relations between L and L^* one gets
$$(L^*L - LL^*)\xi = (n - p - q)\xi,$$
and from this one clearly has
$$(n - p - q)(\xi, \xi) \leq 0,$$
if $\xi \in \mathcal{H}^{p,q}(E)$. Now, using the isomorphism between cohomology and their harmonic representatives in this setting, we obtain part 2. of the vanishing theorem.

Now, for the first part, if $E \otimes K^*$ is positive, then $(E \otimes K^*)^* = K \otimes E^*$ is a negative bundle. Thus we have

$$H^q(X, \mathcal{O}(E)) = H^q(X, \mathcal{O}(K \otimes K^* \otimes E)) = H^q(X, \Omega^n(K^* \otimes E)),$$

which is dual to $H^{n-q}(X, \mathcal{O}(K \otimes E^*))$, by Serre Duality. And since this vanishes for $q > 0$ by part 2. we have concluded the proof.

□

Now we are ready to state Kodaira's theorem, which was something conjectured by Hodge.

Theorem 3.7.8. *Let X be a compact Hodge manifold. Then X is a projective algebraic manifold.*

Recall that we are assuming Chow's theorem, and thus the proof reduces to seeing that it can be embedded in projective space.

It follows from this theorem that any compact complex manifold X admitting a positive line bundle $L \to X$ is projective algebraic, since if that is the case, the chern class $c_1(L)$ will have a Hodge form as a representative.

Proof. We are assuming that X is a Hodge manifold, and thus that there is a Hodge form Ω on X.

From this it follows that there is a holomorphic line bundle $E \to X$ such that Ω is a representative for $c_1(E)$, this is (Chap. III Proposition 4.6) in [1]. And thus E is a positive holomorphic line bundle.

We consider μ_0 given by (Chap VI. Proposition 3.3) in [1](one can just think of this μ_0 as the minimum degree of twisting required to produce the construction needed) and take $\mu \geq \mu_0$. With this value of μ we consider $F = E^\mu$. And we consider the vector space of holomorphic sections $\mathcal{O}(X, F) = \Gamma(F)$, which is finite-dimensional due to the finite-dimensionality of cohomology (sections are just H^0).

Our goal now will be to see that there is an embedding of the manifold X into $\mathbb{P}(\Gamma(F))$. To do this we will use a sequence of lemmas to reduce this embedding problem to the vanishing theorem. □

Before we start with the lemmas, we need to do some previous work. Consider the subsheaf of $\mathcal{O} = \mathcal{O}_X$ consisting of germs of holomorphic functions vanishing at p and q, and we denote it by m_{pq}. If $p = q$, then we mean by this the holomorphic functions vanishing to second order at the point p, and we denote it by $m_p^2 (= m_{pp})$, where m_p is the ideal sheaf of germs vanishing to first order at the point p. In this setting we have an exact sequence of sheaves

$$0 \to m_{pq} \to \mathcal{O} \to \mathcal{O}/m_{pq} \to 0,$$

and we can tensor this exact sequence with the locally free sheaf $\mathcal{O}(F)$(sheaf of holomorphic sections of F) to obtain

$$0 \to m_{pq} \otimes_{\mathcal{O}} \mathcal{O}(F) \to \mathcal{O}(F) \to \mathcal{O}/m_{pq} \otimes_{\mathcal{O}} \mathcal{O}(F) \to 0.$$

The quotient sheaf in this sequence becomes

$$\mathcal{O}_p/m_p^2 \otimes_{\mathbb{C}} F_p, \quad x = p = q,$$

$$0, \quad x \neq p,$$

for the case that $p = q$. If $p \neq q$, then we get

$$F_p, \quad x = p$$

$$F_q, \quad x = q$$

$$0, \quad x \neq porq,$$

this comes mainly from the fact that $\mathcal{O}_p/m_p \cong \mathbb{C}$, where m_p is the maximal ideal in the local ring $\mathcal{O}_p$.

We now have a lemma relating this construction to more intuitive objects.

Lemma 3.7.3. *$\mathcal{O}_p/m_p^2 \cong \mathbb{C} \oplus T_p^*X$, and the quotient mapping is represented by*

$$f \in \mathcal{O}_p \to f(p) + df(p).$$

This lemma just follows by expanding f in local coordinates in a power series near the point p.

Now we go back to the sequence

$$0 \to m_{pq} \otimes_{\mathcal{O}} \mathcal{O}(F) \to \mathcal{O}(F) \to \mathcal{O}/m_{pq} \otimes_{\mathcal{O}} \mathcal{O}(F) \to 0,$$

and consider the induced mapping on global sections

$$\mathcal{O}(X, F) \xrightarrow{r} \mathcal{O}_p/m_p^2 \otimes F_p \cong [\mathbb{C} \oplus T_p^*X] \otimes F_p.$$

Now, if f is a local frame for F near p and $\xi \in \mathcal{O}(X, F)$, then we can consider the map

$$r(\xi(f)) = (\xi(f)(p), d\xi(f)(p)) \in \mathbb{C} \oplus T_p^*X,$$

noting that by the choice of a frame $F_p \cong \mathbb{C}$.

Suppose that we knew that this map r were surjective, then we can conclude that we can find sections $\xi_0, ..., \xi_n \in \Gamma(X, F)$, satisfying

$$\begin{aligned} \xi_0(p) &= 1 \\ \xi_j(p) &= 0, j = 1, ..., n \\ d\xi_j(p) &= dz_j, \end{aligned}$$

where the last equality is of course in some local coordinates z_j. From this we deduce that the global sections $\xi_1, ..., \xi_n$, expressed in terms of the frame f, **give local coordinates** for X, in particular,

$$d\xi_1(f) \wedge ... \wedge d\xi_n(f) \neq 0,$$

and moreover $\xi_0 \neq 0$.

Similarly, suppose that the mapping

$$\mathcal{O}(X, F) \xrightarrow{s} F_p \oplus F_q$$

naturally induced from the previous construction is surjective. Then are able to find global sections ξ_1 and ξ_2 satisfying

$$\xi_1(q) = \xi_2(p) = 0$$
$$\xi_1(p) \neq 0$$
$$\xi_2(q) \neq 0.$$

And with this we are very close to building the desired embedding. The following result formalizes this idea.

Lemma 3.7.4. *If the mappings r, s above are surjective for all points $p, q \in X$, then there exists a holomorphic embedding of the manifold X into $\mathbb{P}^m$, where $\dim_{\mathbb{C}} \mathcal{O}(X, F) = m + 1$.*

The reader interested in seeing the proof is referred to [1] (Chap. VI Lemma 4.3). Note that the proof formalizes the idea above, note that one also has to check, by computing the Jacobian determinant, that one is constructing in fact an embedding.

Finally there is just one more lemma that completes the proof. Since we know from above that if r, s are surjective (for all points), then the proof is completed, then one would like this to be the case. And in fact this is what happens.

Lemma 3.7.5. *The mappings r, s above are surjective for all points $p, q \in X$.*

The reader interested in the concrete details of this proof is also referred to [1] (Chap. VI Lemma 4.4). We omit the proof since it requires the theory of divisors, of which we have not talked at all during the work. We note to the reader that the fundamental role of the vanishing theorem in the proof is to show, via constructing certain exact sequences and diagrams relating them, that a certain cohomology group is zero.

And this fact completes the proof of the Kodaira Embedding theorem.

3.8 Kähler Einstein metrics and Calabi-Yau manifolds

In this last section on complex differential geometry, we will study a special class of metrics on Kähler manifolds, which are called *Kähler-Einstein metrics*. There has been a lot of work in this metrics from both the complex-analytic perspective and the algebraic perspective, and we will not even atempt to cover all such bast material.

We will focus in a deep problem that motivated a lot of development during the second half of last century, and also contributed to a branch of High Energy physics known as *String theory.* Although we will not say much about the connections to physics, it is important to know that there are some, and that investigating them is still even today an open problem. There are many important observations that physicists have made during the years by studying those manifolds, and that have turned to be important in the mathematical arena, one such topic is what is usually called *Mirror Symmetry*, which has contributed to a lot of development on the mathematical community. Hence the topic of Kähler-Einstein manifolds, and a special kind of manifold of which we will talk later, the so-called *Calabi-Yau manifolds* is a very rich topic both from the mathematical and physical points of view, and has been a common ground for mathematicians and physicists. During this section we will assume that our manifold M is connected.

The main geometric object during this section will still be that of Kähler manifolds. But now we would like to look at Kähler manifolds from "the real perspective" and view them as an even dimensional smooth manifold M, together with an integrable almost-complex structure J and a Kähler metric g, with the associated fundamental form. We will specially be interested in the associated Ricci form, that we will denote by $\rho_{ac} = J_a^b R_{bc}$. We saw on previous sections that ρ is a closed $(1,1)$-form and that furthermore, at the cohomology level we have that $[\rho] = 2\pi c_1(M) \in H^2(M,\mathbb{R})$. The natural question now to ask is, for a given manifold, which closed $(1,1)-$forms are actually coming from a Kähler metric on M. This question was posed by the Italian mathematician Eugenio Calabi on 1954, in the following form:

Calabi Conjecture: *Let (M,J) be a compact, complex manifold and g a Kähler metric on M with associated Kähler form ω. Given ρ' a real, closed $(1,1)-form$ on M satisfying $[\rho'] = 2\pi c_1(M)$. Then there is a unique Kähler metric g' on M with associated Kähler form ω' satisfying $[\omega'] = [\omega] \in H^2(M,\mathbb{R})$, and whose Ricci form is ρ'.*

This conjecture, although posed in 1954, when Calabi also showed that if the metric g' exists, then it must be unique, remain unsolved for more than 20 years. It was finally proven by the chinese mathematician Shing-Tung Yau in 1976. Before Yau's result, several attempts were made at solving this conjecture, but none of them seem to be able to solve it completely.

This result has an important consequence, although it may be overlooked at first sight, and it is the case where our manifold has vanishing first Chern class, meaning $c_1(M) = 0$. If this is the case, one can choose as ρ' the zero form, since it will satisfy the hypotheses. Thus the conjecture (now theorem) implies that we can find a (unique) Kähler metric such that it's Ricci form is just ρ', and that means zero. Hence we have seen that if the conjecture holds true, we can construct families of *Ricci-flat Kähler metrics* on compact Kähler manifolds, namely in those having $c_1(M) = 0$. This manifolds, due to this conjecture have a special name in the literature, and due to the two main man that contributed to their study, they are called *Calabi-Yau manifolds*. This manifolds turn out to play a very important role in String theory, and is the original setting where physicists started to "see" Mirror Symmetric manifolds, but more on that later.

We will not give a complete proof of the conjecture, but we will try to sketch the main points that are involved in proving the results, to see what kind of techniques can be used to prove a result like this.

Reformulation as a PDE: We will first reformulate the Calabi conjecture as a nonlinear elliptic PDE in a real function ϕ. We will be in the setting described by the conjecture above, and thus we have a compact complex manifold (M, J) (of real dimension $2m$) with Kähler metric g that has ρ as its associated Ricci form. If ρ' is a real, closed $(1,1)-$form with $[\rho'] = 2\pi c_1(M)$. Solving the Calabi conjecture amounts to finding a Kähler metric g' with Kähler form ω' satisfying $[\omega] = [\omega']$ and whose Ricci form is just ρ'.

Since we have that $[\rho'] = 2\pi c_1(M) = [\rho]$, this means that we have $[\rho' - \rho] = 0$ in $H^2(M, \mathbb{R})$. This means, by (LEMMA ddc) that we can find a real (smooth) function f on our manifold M, unique up to constant, such that

$$\rho' = \rho - \frac{1}{2} dd_c f,$$

recall the (real) operator $d_c = i(\overline{\partial} - \partial)$, that we introduced previously while studying Kähler manifolds.

Now we define a smooth positive function F by requiring that $(\omega')^m = F\omega^m$.

We can deduce, by using some identities on Kähler manifolds, that

$$\frac{1}{2} dd_c(\log F) = \rho - \rho' = \frac{1}{2} dd_c f,$$

and hence we conclude that $dd_c(f - \log F) = 0$, and thus that $f - logF$ must be a constant on M.

If we now define a constant $A > 0$ by requiring that $f - logF = -logA$. This just amounts to saying that $F = Ae^f$, and thus ω', the fundamental form associated with g', must satisfy

$$(\omega')^m = Ae^f \omega^m.$$

Since we are seeking for solutions requiring $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$, and we are assuming that M is compact, we deduce that

$$\int_M (\omega')^m = \int_M \omega^n,$$

and if we substitute the previous relationship here we obtain

$$\int_M (\omega')^m = A \int_M e^f dV_g = \int_M dV_g = vol_g(M),$$

where dV_g stands for the volume form on our manifold M induced by the metric g, and $vol_g(M)$ is just the total volume with respect to this same metric. By this last equation, we have determined the constant A provided that we are able to find the function f, the other function $F = Ae^f$ is also deduced from this information.

And hence we have seen an equivalence between the previous statement of the Calabi conjecture and the following:

Calabi Conjecture (v2): *Let (M, J) be a compact, complex manifold and g a Kähler metric on M with associated Kähler form ω. Consider f to be an arbitrary smooth real function on M, and define $A > 0$ by $A \int_M e^f dV_g = vol_g(M)$. Then there exists a unique Kähler metric g' (on M) with associated Kähler form ω' satisfying*

$$[\omega'] = [\omega] \in H^2(M, \mathbb{R})$$
$$(\omega')^m = Ae^f \omega^m.$$

To understand this reformulation, and why is it even useful, we shall look a bit into it. Firstly, this conjecture is about finding metrics with prescribed volume forms, and as we saw this is equivalent to the original conjecture. And every volume form on our manifold M can be written as FdV_g, for a certain smooth real function F. To this form we are imposing thus two conditions, the first is just positivity, meaning that we require $F > 0$, and secondly that it has the same total volume as the original volume form of the manifold, meaning $\int_M FdV_g = \int_M dV_g$. Then the conjecture just says that there is a unique Kähler metric g' such that the associated form is in the same Kähler class ($[\omega'] = [\omega] \in H^2(M, \mathbb{R})$) and with the prescribed volume form, meaning $dV_{g'} = FdV_g$.

Although it may not seem to be the case, this is a big simplification if we compare it to the (equivalent) original formulation of the conjecture, but why so? The Conjecture, as originally posed, prescribes the Ricci curvature of the metric g', which has a dependence both on the metric g' and on it's second derivatives! In effect on the original conjecture requires one to solve m^2 equations on the metric g'. But now note that in this reformulation (Conjecture v2), the statements depends only on the metric g', but it **doesn't deppend on its derivatives!**, and in fact is just a single real equation on the metric g'. One can deduce thus, that this second reformulation in fact is a big step, since we have reduced hugely the number of equations for our metric, and in this sense the conjecture is hugely simplified by the observations that were made before.

Now we want to reformulate it (yet again) to obtain the final form of the conjecture, the one that actually was proved by Yau once this whole reformulation was done, and where we will see explicitly how solving the conjecture amounts to solving the nonlinear elliptic PDE we said before, so let's see what is the PDE. To see how to reformulate the conjecture again we start by noting that, since we are requiring that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$, this means that we can find a function $\phi \in \mathcal{C}^\infty(M)$ (unique up to a constant) such that we can relate both forms by the equation

$$\omega' = \omega + dd_c\phi.$$

Since we do still have left one degree of freedom for the function ϕ (it is unique up to constant), we can impose one additional constraint on this function, and we will impose that ϕ satisfies

$$\int_M \phi dV_g = 0,$$

this thus specifies our function ϕ uniquely, and hence we have reformulated the Calabi Conjecture as follows:

Calabi Conjecture (v3): *Let (M, J) be a compact, complex manifold and g a Kähler metric on M with associated Kähler form ω. Consider f to be an arbitrary smooth real function on M, and define $A > 0$ by $A\int_M e^f dV_g = vol_g(M)$. Then there exists a unique function $\phi \in \mathcal{C}^\infty(M)$ satisfying*

1. *$\omega + dd_c\phi$ is a positive $(1,1)-$form. This is just the ω' in our previous formulation.*
2. *$\int_M \phi dV_g = 0$. As we saw, we impose this equation to get uniqueness for ϕ.*
3. *$(\omega+dd_c\phi)^m = Ae^f\omega^m$. And this is the equation we had on the previous reformulation, relating both ω and ω'.*

Now, due to the fact that the highest power of our Kähler forms (as with any differential $2-$form) is determined by a multiple of the volume form, and essentially only depends on the determinant of the metric (and maybe some factorials deppending on the normalization chosen), one can see that part (3.) of the previous theorem can be reformulated. To do that we shall look again at the equation (part 1.)

$$(\omega + dd_c\phi)^m = Ae^f\omega^m,$$

now, due to the fact that, we are requiring that the metric g' that we want to find to solve the conjecture has to satisfy

$$\omega' = \omega + dd_c\phi.$$

If we put those equations together and look at them (at the level of the metric) in local holomorphic coordinates $z_1, ..., z_n$ on a certain open set, then this last equation tells us that

$$g'_{\alpha\overline{\beta}} = g_{\alpha\overline{\beta}} + \partial_\alpha\overline{\partial}_{\overline{\beta}}\phi.$$

Hence, we have seen that equating the top level (associated) Kähler forms, just amounts to requiring the following:

Choosing holomorphic coordinates $z_1, ..., z_n$ on $U \subset M$ open set, the metric $g_{\alpha\overline{\beta}}$ may be interpreted as a $m \times m$ Hermitian matrix. And part (3.) of the Calabi Conjecture (v3) is equivalent to imposing that the function ϕ satisfies

$$\det(g_{\alpha\overline{\beta}} + \frac{\partial^2 \phi}{\partial z_\alpha \partial \overline{z}_{\overline{\beta}}}) = Ae^f \det(g_{\alpha\overline{\beta}}).$$

As we just saw this is equivalent to requiring that the top exterior forms ω' and ω satisfy the equation

$$(\omega')^m = Ae^f \omega^m,$$

since being top exterior forms, they are just related by their coefficient, and this comes from the determinant of the metric. Hence we have reformulated (yet again) the conjecture, and we will state it in this form just to make clear the result that we want to sketch, but recall that all these versions that we are describing are just **equivalent formulations** of the conjecture. This means that one could in principle at solving the conjecture from any of those formulations, it will turn out that this last reformulation is the most amenable to prove, and in fact it was the one proven by Yau.

Calabi Conjecture (v4): *Let (M, J) be a compact, complex manifold and g a Kähler metric on M with associated Kähler form ω. Consider f to be an arbitrary smooth real function on M, and define $A > 0$ by $A\int_M e^f dV_g = vol_g(M)$. Then there exists a unique function $\phi \in \mathcal{C}^\infty(M)$ satisfying*

1. *$\omega + dd_c\phi$ is a positive $(1,1)-$form. This is just the ω' in our previous formulation.*
2. *$\int_M \phi dV_g = 0$. As we saw, we impose this equation to get uniqueness for ϕ.*
3. *Choosing holomorphic coordinates $z_1, ..., z_n$ on $U \subset M$ open. Then the condition on ϕ is*
$$\det(g_{\alpha\overline{\beta}} + \frac{\partial^2 \phi}{\partial z_\alpha \partial \overline{z}_{\overline{\beta}}}) = Ae^f \det(g_{\alpha\overline{\beta}}).$$

And this will be the last reformulation we make of the conjecture, but now we will see why this reformulation is indeed useful. First, note that we have "reduced" a highly non-trivial problem in complex geometry to a nonlinear, second order PDE in the function ϕ. Now if one wants to solve the conjecture, one just has to show that a certain PDE has a unique, smooth solutions. This doesn't mean that it will be easy, but at least one has all the techniques of PDE's and Functional Analysis to use to try to solve this. One should note that, although the reformulation may seem to tell us that this is just a problem in analysis, this is in fact a **very difficult problem in analysis**, this is due to the fact that in general nonlinear equations are not that easy to solve, and the nonlinearities in our equation are not specially "easy" to deal with, since they are nonlinear in the derivatives of highest order, making it a hard problem. This is mainly the reason why it took more than 20 years to be solved, since although reformulated as a PDE, the PDE that comes out is highly non-trivial.

One should note, as an historical remark, that these types of PDE's are not the first time that pop out in mathematics, and in fact have a special name, they are called *Monge-Ampère equations*. These type of equations usually arise in differential geometry, and have been a common ground for solving many questions in this field. These kind of equations were first studied by two french mathematicians and physicists: Gaspard Monge, one of the fathers of differential geometry, and André-Marie Ampère, one of the founders of classical electromagnetism. Both started their study of this equations, since they appeared in their respective studies. Afterwards, this kind of equations, due to their importance in many fields of mathematics and physics, have been intensively studied by many mathematicians and physicists, such as: Bernstein, Pogorelov, Fefferman, Niremberg...

One should note here that in fact that, in the last reformulation of the theorem, part (1.) actually follows from (3.), as can be seen by the following lemma, that we will not prove, since we just want to sketch the proof and we will not deal with a lot of the technical details that are needed at every step, although we will make some mention to them if we find it enlightening. The interested reader can check the proof in [2]

Lemma 3.8.1. *Let (M, J) be a compact Kähler manifold with metric g and Kähler form ω, $f \in \mathcal{C}^0(M)$ and a constant A defined by $A\int_M e^f dV_g = vol_g(M)$. If $\phi \in \mathcal{C}^2(M)$ satisfies the equation*

$$(\omega + dd_c\phi)^m = Ae^f\omega^m,$$

then the $(1,1)-$form $\omega + dd_c\phi$ is positive.

With this lemma under our belt, we can clearly see that the Calabi conjecture is just equivalent to solving the PDE we saw before. We now want to see how Yau was able to deal with this problem, we will begin by stating 4 theorems that are needed during the proof, we will briefly comment about what they mean to understand the role they play in the proof. As with the previous lemma we will not provide an explicit proof of this theorems, and the interested reader is referred to Chapter 5 in [2], where the proofs of these results are given. Once we understand the significance of the results, we will be able to start sketching the actual proof of the conjecture. So here are the theorems that are a fundamental part of the proof given by Yau. In all of these theorems (M, J) will be as usual a complex Kähler manifold with Kähler metric g and associated form ω.

Theorem 3.8.1. *Let $Q_1 \geq 0$, then there exist some positive constants $Q_2, Q_3, Q_4 \geq 0$, that only depend on M, J, g and the constant Q_1 such that the following result holds: If*

$f \in \mathcal{C}^3(M), \mathcal{C}^5(M)$ and $A > 0$ satisfy

$$\|f\|_{\mathcal{C}^3} \leq Q_1; \int_M \phi dV_g = 0; (\omega + dd_c\phi)^m = Ae^f \omega^m,$$

then we have the bounds

$$\|\phi\|_{\mathcal{C}^0} \leq Q_2; \|dd_c\phi\|_{\mathcal{C}^0}; \|\nabla dd_c\phi\|_{\mathcal{C}^0} \leq Q_4.$$

Remark. This important theorem was due to Yau, and was one of the hardest problems to deal with while solving the conjecture. In the mathematics literature, this type of bounds are sometimes called *a priori estimates*, meaning that one can find in advance that arbitrary solutions to a certain equation must satisfy certain bounds. And this is important since one can thus restrict the (Banach) space where one is looking for solutions to a certain subspace that may have some interesting special properties. Since we want to show that there are solutions to the equation which are unique, this estimates (a priori) play a fundamental role in the argument. This technique of a priori estimates was introduced into math by the Russian/Soviet mathematician Bernstein, who devoted to the study of Dirichlet's boundary problem for non-linear equations of elliptic type, recall that precisely that it the type of PDE that we are dealing with.

Before stating the theorem, we define $\mathcal{C}^{k,\alpha}$ to be the set of $f \in \mathcal{C}^k$ whose *Hölder norm* $\|f\|_{\mathcal{C}^{k,\alpha}} = \|f\|_{\mathcal{C}^k} + [\nabla^k f]_\alpha$ is finite. The reader not familiar with this notions can just think of $\mathcal{C}^{k,\alpha}(M)$ as the space of functions that are "$(c+\alpha)$" times differentiable.

Theorem 3.8.2. *Let $Q_1, ..., Q_4 \geq 0$ and $\alpha \in (0,1)$. Then there exists another positive constant $Q_5 \geq 0$, depending only on $M, J, g, Q_1, ..., Q_4$ and α such that the following result holds: If $f \in \mathcal{C}^{3,\alpha}(M), \phi \in \mathcal{C}^5(M)$ and $A > 0$ satisfy both $(\omega + dd_c\phi)^m = Ae^f\omega^m$ and the estimates*

$$\|f\|_{\mathcal{C}^{3,\alpha}} \leq Q_1; \|\phi\|_{\mathcal{C}^0} \leq Q_2; \|dd_c\phi\|_{\mathcal{C}^0}; \|\nabla dd_c\phi\|_{\mathcal{C}^0} \leq Q_4,$$

Then we have that $\phi \in \mathcal{C}^{5,\alpha}$ and it's norm is bounded as $\|\phi\|_{\mathcal{C}^0} \leq Q_3$. If we assume more regularity, namely that $f \in \mathcal{C}^{k,\alpha}(M)$ for some $k \geq 3$, then $\phi \in \mathcal{C}^{k+2,\alpha}(M)$, and if $f \in \mathcal{C}^\infty(M)$ then we have $\phi \in \mathcal{C}^\infty$.

Remark. This result uses deeply the big amount of theory that has been developed about properties of solutions of elliptic equations, and it basically deals with the regularity properties that we want our function to have. Although we do not provide a proof of this theorem, we think it is important for the reader to know what kind of techniques are involved in their proof. This is mainly an analytical result, and builds deeply on some results about differentiability of solutions to elliptic equations.

In the following theorem we note that both f' and ϕ' will be arbitrary functions, they are not the derivative.

Theorem 3.8.3. *If we fix a parameter $\alpha \in (0,1)$ and suppose that we have $f' \in \mathcal{C}^{3,\alpha}(M), \phi' \in \mathcal{C}^{5,\alpha}(M)$ and a constant $A' > 0$ satisfying*

$$\int_M \phi' dV_g = 0; (\omega + dd_c\phi')^m = Ae^{f'}\omega^m.$$

Then for an arbitrary function $f \in \mathcal{C}^{3,\alpha}(M)$ being close to f', meaning that the norm $\|f - f'\|_{\mathcal{C}^{3,\alpha}}$ is sufficiently small, then we have a function $\phi \in \mathcal{C}^{5,\alpha}(M)$ and a constant $A > 0$ satisfying

$$\int_M \phi dV_g = 0 and (\omega + dd_c\phi)^m = Ae^f\omega^m.$$

Remark. In this theorem what one understands is the behaviour of the solutions to the equation in the sense of Sobolev spaces, meaning that we can solve the problem again once we have a solution (in the Sobolev space) if one requires that the functions f and f' are "close" to each other, meaning that their Sovolev norm is small.

Theorem 3.8.4. *If $f \in \mathcal{C}^1(M)$, then there is at most one function $\phi \in \mathcal{C}^3(M)$ and a constant $A > 0$ such that*

$$\int_M \phi dV_g = 0 and (\omega + dd_c\phi)^m = Ae^f\omega^m.$$

Remark. In this theorem the question that is answered is clearly the question of the uniqueness of the solution to our problem, meaning that if it exists, there is no more than one. This was proved by Calabi before the groundbreaking work of Yau and is more accessible as a result than the previous theorems.

In a nutshell, this theorems tells us about two properties of the solution that we are trying to find, the first and the third theorem deal with the actual *existence* of the function that we want to find. The second theorem deals with the *regularity/smoothness* that this function is going to have, and depending on the regularity that we desire we know that we have to impose some concrete restrictions. And finally the fourth theorem deals with the *uniqueness* of the possible solution.

Now what we will see is how this facts actually combine together in the proof of the theorem, and once we start developing the scheme of the proof, that is more intuitive than what one might think a priori (although there are many analytic details of course), we believe that the role and the relevance that this theorems have will come clear.

But what is the idea behind the proof? It is just using a "simple" tool that mathematicians have known for centuries and that has proven to be very influential in a lot of fields of mathematics, both applied and theoretical, to obtain results about solutions to certain equations. This tool is what is usually known as *the continuation method.* This method tries to solve a particular kind of equation by deforming it into a simpler one and later transporting this easier solution to the original equation. In our case the goal is to prove that certain nonlinear equation, namely

$$(\omega + dd_c\phi)^m = Ae^f\omega^m,$$

has a solution ϕ. The first step in the continuation method is to find a similar equation to the one we have but that is more ameanable to analysis and easier to find a concrete solution, in our case we choose

$$(\omega + dd_c\phi)^m = \omega^m,$$

meaning that we have "forgoten" about the exponential term (and the constant A). But why is this equation easier? Simple, because if one looks carefully, this equation has the trivial solution $\phi = 0$. And hence we have completed the first step of the continuation, we found a related equation having a solution, and in this case a very simple solution.

What one does now is to construct, in a sense, a path between the solution that we know and the desired one. In more concrete mathematical terms this means that we want to find a 1-parameter family of equations depending continuously on a certain parameter $t \in [0, 1]$ such that when $t = 0$ the equation is just the "simplified version" of which we know a solution, and for $t = 1$ the equation is the one we are actually interested in solving. The concrete family for our concrete problem is

$$(\omega + dd_c\phi_t)^m = A_t e^{tf}\omega^m.$$

Note that in this family, if we take $t = 0$, we have the equation with trivial solution $\phi = 0$, and if we take $t = 1$, we obtain the equation that we want to solve. Now the power of continuation is to be able to go from one to the other. To do this, one must see that the set S of values of t for which there is a solution is both open and closed. Since the interval is connected this will mean that either $S =$ or $S = [0, 1]$. And due to the fact that we have a solution for $t = 0$, this means that S will be the whole interval, thus having a solution for $t = 1$ for the original equation.

And this is the basic idea behind the proof, once one has settled up all the technical analytical details (which are of fundamental importance) by using this continuation Yau was able to solve the conjecture. We are now going to see how to formalize the ideas that we have given about the continuation method in our concrete case, particularly we will show the proofs of the set S being open and closed, and how this allowed Yau to finish the proof. We begin with a definition that will remind us of the continuation idea we have talked about.

Definition 3.8.1. Let (M, J) be a compact complex manifold with Kähler metric g and associated form ω. Fixing $\alpha \in (0, 1)$ and $f \in \mathcal{C}^{3,\alpha}(M)$, we define the set S to be the set of all possible values of $t \in [0, 1]$ for which there is a function $\phi \in \mathcal{C}^{5,\alpha}(M)$ with vanishing integral, meaning $\int_M \phi dV_g$, and a constant $A > 0$ such that we can find a solution

$$(\omega + dd_c\phi)^m = Ae^{tf}\omega^m.$$

This is of course very similar to the idea that we just saw before, just note that to appropriately use the results of the theorems one is required to work with the spaces $\mathcal{C}^{\cdot,\alpha}$, but that is just a technical requirement for the proof to hold, the idea behind it is the same, which was what we wanted to emphasize before.

Now, as we said, we want to see that the set S is both open and closed. We will start by seeing why is it closed, and this will mainly be a consequence of the first two theorems Yau introduced into the proof. To see that the set is closed what we want to see is that it contains its limit points. The main analytical resource that we need here to do this is the notion, introduced before, of the *a priori estimates*. Under this setting, one usually (in the continuity method) wants to see that the associated solutions for each time lie all in a *compact* subset of some Banach space. If this is the case, there will be a convergent subsequence that has a limit point, and one can show that this is the solution for the desired limit point. This is the usual scheme to show closedness while applying the continuity method. We will see now how this is formalized for our particular case of interest.

Theorem 3.8.5. *In the previous definition, the set S is a* ***closed*** *subset of the interval* $[0, 1]$.

Proof. As we said, we have to see that S contains its limit points. So we take an arbitrary sequence $\{t_j\}_{j=0}^{\infty}$ in S converging to some $t' \in [0, 1]$ (it is clear that the point will lie on the interval since the whole interval is a closed set). We want to see that actually the limit point t' belongs to the set S. Now, since for every $t_j \in S$ in our sequence, by definition we have an associated function $\phi_j \in \mathcal{C}^{5,\alpha}$ and a constant $A_j > 0$ satisfying

$$\int_M \phi_j dV_g = 0 \quad \text{ and } \quad (\omega + dd_c\phi_j)^m = A_j e^{t_j f} \omega^m.$$

Now we will make use of the previously mentioned theorems to see that the limit point is also in S. To do this we start by defining the constant $Q_1 = \|f\|_{\mathcal{C}^{3,\alpha}}$. From the first theorem we get some constants Q_2, Q_3, Q_4 which depend on Q_1 and from the second theorem we get another constant Q_5 which depends on $Q_1, ..., Q_4$.

Since we have that $t_j \in [0, 1]$, this means that we have the bound

$$\|t_j f\|_{\mathcal{C}^3} \leq \|t_j f\|_{\mathcal{C}^{3,\alpha}} \leq \|f\|_{\mathcal{C}^{3,\alpha}} = Q_1,$$

where the first inequality just comes from the definition of the $\mathcal{C}^{3,\alpha}$ norm and the other inequality from the fact that t_j is smaller (or equal) than 1, and thus by the properties of the norm in a Banach space we deduce the bound.

We now apply the first theorem to obtain a series of *a priori estimates* for the solutions

ϕ_j. More concretely, what one obtains is:

$$\begin{aligned}\|\phi_j\|_{\mathcal{C}^0} &\le Q_2\\ \|dd_c\phi_j\|_{\mathcal{C}^0} &\le Q_3\\ \|\nabla dd_c\phi_j\|_{\mathcal{C}^0} &\le Q_4.\end{aligned}$$

We here note the important fact that, for every j, the bounds are the same! The constants do not depend on the particular value of j, and hence are uniform in j, as one can see by looking at the theorem.

Thus, we can use the second theorem to obtain now that $\phi_j \in \mathcal{C}^{5,\alpha}(M)$ with the bound: $\|\phi_j\|_{\mathcal{C}^{5,\alpha}} \le Q_5$, and as before, this holds for every j. Meaning that as before we have a uniform bound in j. Now we can use a theorem (EXPLICARLO). With this we get that the inclusion

$$\mathcal{C}^{5,\alpha}(M) \to \mathcal{C}^5(M)$$

is a *compact* map. With this fact in our hands, and knowing that we have a uniform bound for ϕ_j, meaning that in the space $\mathcal{C}^{5,\alpha}$ the sequence $\{\phi_j\}$ is bounded. Due to the compacity of the map, we deduce from this that the sequence $\{\phi_j\}$ lies in a compact subset of $\mathcal{C}^5(M)$. Since we have a sequence $\{\phi_j\}$ inside of a compact subset, this means that there is a subsequence $\{\phi_{n_j}\}$ which is convergent in $\mathcal{C}^5(M)$. We denote by ϕ' the limit of this subsequence. Now we want to see that in fact this limit satisfies the requirements we want. To do this, we define a constant $A' > 0$ by imposing that it satisfies

$$A' \int_M e^{t'f} dV_g = vol_g(M).$$

Now, since we have the associated subsequence of parameters $t_{n_j} \to t'$, since it is just a subsequence of the original convergent sequence, it is clear, by an usual argument of dominated convergence, that we must have that the associated constants converge to this A', meaning that

$$A_{n_j} \to A' \quad asj \to \infty.$$

Since we have that our subsequence $\{\phi_{n_j}\}$ is convergent in $\mathcal{C}^2(M)$ (due to it's convergence in $\mathcal{C}^5(M)$) we can take the limit of

$$\int_M \phi_{n_j} dV_g = 0 \quad and \quad (\omega + dd_c\phi_{nj})^m = A_{nj} e^{t_{nj} f} \omega^m,$$

as $j \to \infty$, and thus obtaining the desired result, namely that ϕ' satisfies

$$\int_M \phi' dV_g = 0 \quad and \quad (\omega + dd_c\phi')^m = A' e^{t'f} \omega^m.$$

Now by the first two theorems due to Yau, we obtain that $\phi' \in \mathcal{C}^{5,\alpha}(M)$. And therefore we have seen that the limit value $t' \in S$. And thus the set S contains it's limit points, and is thus closed, as we wanted to see. $\square$

Hence, now we now that our set S is closed. Now we are going to see that it is also open.

Theorem 3.8.6. *The set S is an open subset of* $[0,1]$.

Proof. Start with an arbitrary point $t' \in S$, we want to see that the set S contains some open neighborhood around t'. By the definition of S, we have that there exists a function $\phi \in \mathcal{C}^{5,\alpha}(M)$ with $\int_M \phi' dV_g = 0$ and a constant $A' > 0$ such that

$$(\omega + dd_c\phi')^m = A'e^{t'f}\omega^m.$$

Now we will apply the third theorem due to Yau, with $t'f$ in place of f' and tf in place of f, for $t \in [0,1]$. By the theorem we know that if $|t - t'| \, \|f\|_{\mathcal{C}^{3,\alpha}}$ is sufficiently small (namely when t and t' are very close) then there exists a function $\phi \in \mathcal{C}^{5,\alpha}(M)$ and a constant $A > 0$ such that

$$\int_M \phi dV_g = 0 \quad \textit{and} \quad (\omega + dd_c\phi)^m = Ae^{tf}\omega^m.$$

By the definition of the set S, this just means that $t \in S$. Thus we have seen that for values $t \in [0,1]$ that are sufficiently close to the original t' then $t \in S$, this means that S contains some open neighborhood in $[0,1]$ for each value $t' \in S$. And thus we have seen that the set S is open. □

Now we want to sum all the information we have into the final proof of the conjecture, in the version that we are using to sketch the proof, but in more concrete terms, meaning that the result in fact will only require some certain degree of regularity. We do not need functions to be $\mathcal{C}^\infty$, it also works (similarly) under fewer assumptions on the regularity, which may be useful if one is interested in doing differential geometry with $\mathcal{C}^k$ functions. Later we will see how to finish the proof when we actually desire to obtain a smooth unique function, but this will just follow from theorems 2 and 4 given by Yau.

Theorem 3.8.7. ***Calabi Conjecture (less regularity):*** *Let (M, J) be a compact, complex manifold and g a Kähler metric on M with associated Kähler form ω. Take a value $\alpha \in (0,1)$ and a function $f \in \mathcal{C}^{3,\alpha}(M)$. Then there exists a function $\phi \in \mathcal{C}^{5,\alpha}$ and a positive constant $A > 0$ such that the following statements hold:*

1. *$\omega + dd_c\phi$ is a positive $(1,1)-$form.*
2. *$\int_M \phi dV_g = 0$.*
3. *$(\omega + dd_c\phi)^m = Ae^f\omega^m$.*

Proof. As we said before, this proof follows the continuation method scheme, and we will use the notation from the above results. Due to the previous two results, we know that the set S, of values of t for which we can find a solution is both open and closed. Obviously since $[0,1]$ is connected this implies that S is either empty or the whole interval. To see that it is the whole interval thus we just have to see that for some value of t there is a solution. But, as we said before, this is clear since for $t = 0$ we have the trivial solution $\phi = 0$. And thus this means that $0 \in S$, and hence $S = [0,1]$. By the definition of S, if we take $t = 1$, this means that there is a function $\phi \in \mathcal{C}^{5,\alpha}(M)$ with $\int_M \phi dV_g = 0$ and satisfying the equation

$$(\omega + dd_c\phi)^m = Ae^f\omega^m,$$

where $A = A_1$. Hence we have seen that both parts (2.) and (3.) of the theorem hold, and by a lemma that we saw previously, this means that part (1.) also holds. And this completes the proof. $\square$

Remark. Now, what about the smooth case? What if our original function $f \in \mathcal{C}^\infty(M)$? Using this theorem we can deduce the existence of $\phi \in \mathcal{C}^{5,\alpha}(M)$ and of a constant $A > 0$ satisfying the conditions of the Calabi conjecture. But now, can we say that ϕ is actually smooth? In fact, this is guaranteed by the second theorem due to Yau, which gives us that the function ϕ is actually a $\mathcal{C}^\infty(M)$ function provided that the original function f is also $\mathcal{C}^\infty$. Thus we have the smoothness of the solution, and the uniqueness clearly just comes from using the Yau's Theorem 4, which deals precisely with that uniqueness. Hence we have now proven the complete Conjecture also in the $\mathcal{C}^\infty$ case, which is the usual setting under which differential geometers work, although not always, and that is the reason that it is useful to have similar results with fewer conditions on the regularity.

And so, we have finished the sketch we wanted to give about how the Calabi conjecture was solved by Yau, and what techniques were involved in the solution. Clearly a complete proof has to account for the missing details that we omitted, namely the 4 theorems that Yau proved, and the intermediate lemma that we didn't prove. The lemma is relatively easy to prove, as one can see in (REFERENCIA), but the theorems (mainly 1,2 and 3) are the hardest part of the proof, and the main reason that the proof took so long. Since as is more or less common knowledge in the mathematical and physical circles, dealing with non-linear PDE's is not at all an easy task, neither theoretically nor practically. But one should note that the work of Yau is not only restricted to the solution of this conjecture, for which he obtained the Fields Medal in 1982, but has also worked in several areas of differential geometry that are very closely related to physics. Some of his main results are (apart from the Calabi conjecture) the *positive energy theorem* in the field of General Relativity, and also for proposing a program to better understand the phenomena of *Mirror Symmetry*, where he together with other mathematicians proposed the *SYZ conjecture*, possibly allowing for the explicit construction of the mirrors. Obviously a single paragraph can not make justice to such a great mathematician, and thus the interested reader is highly encouraged to go and read about his mathematical and physical works. His developments are nowadays a fundamental tool both for differential geometers and specially String theorists. We will now want to devote a bit of time to explaining why this conjecture has anything to do with physics (since a priori it doesn't seem like), the role that so-called *Calabi-Yau manifolds* play in physics and some examples of them.

Recall that we defined *Calabi-Yau manifolds* as those compact Kähler manifolds whose first Chern class vanishes. Clearly by the Calabi conjecture, this means that we can find a Kähler metric in the same class with vanishing Ricci curvature. Simple and familiar examples are the 1-dimensional (complex) compact manifolds satisfying this requirements, and these are precisely Elliptic curves, which are a geometric object of fundamental importance for many areas of mathematics and physics. And there are more examples on higher dimensions. But what do this notions have to do with physics? Where is the relationship? The answer to this questions is to be found in *String theory*, a branch of modern theoretical physics that has had a profound impact on the physical community. String theory, more than a concrete theory, is a framework where the usual point-like particles that are usually studied in particle physics, for instance in the Standard Model, are replaced by one-dimensional objects that are called *strings*, hence the name of the theory. The purpose of this theory is to understand the behaviour and propagation of this strings through space-time both semi-classically (as the physicists say) or at the quantum level. The point of view of string theory is that those strings, although at their own scale look like little strings propagating, but at scales which are larger than the string scale, it looks just like an ordinary particle with properties like mass, charge, spin, and other physical properties which are determined by the vibrational state of the string. One of the main reasons that string theory has become one of the most important theories in theoretical physics nowadays is because the strings can also describe the gravitons, the quantum mechanical particles that carry gravitational force. Thus the development of string theory has been mainly motivated due to the fact that many physicists believe that it may be the correct description of Quantum Gravity, one of the deeper and hardest problems of contemporary theoretical physics.

One of the fundamental ideas that is used to build many String theory models is the idea of *SuperSymmetry* (often called *SUSY*), which is a postulated symmetry between bosons (force carriers) and fermions (matter). It has not been observed empirically at any particle accelerator in the world, and this is a big problem that the String community is tackling nowadays. As with many aspects of String theory, from a physicist perspective, it has not been at all experimentally checked, and thus there are many negative criticisms to the theory by other physicists. And there is a really big debate in the theoretical physics community nowadays about this, since we do not have still a theory of Quantum Gravity. And one of the main problems is that experimentally we are really far from reaching the energy levels that could actually test those theories (in fact any other theory of Quantum Gravity has the same problem). And thus there is little experimental evidence for string theory, but many physicists still believe that, although it turns out to be wrong experimentally, it has produced a number of fundamental results that can not be neglected. From a purely mathematical perspective, the story is a bit different, since the fact that the theory ends up describing the fundamental interactions of nature is something that may or may not appeal to some mathematicians, but the bulk of work done by physicists in the area is so big and full of mathematical observations that we can not look away. String theory is one of the areas of physics where many of the developments of modern mathematics have been applied more successfully, differential geometry, complex geometry, algebraic geometry, and many more areas of math are used in String theory. To deal with many of the quantum phenomena, they usually use what are usually called (Feynman) *Path Integrals*, an infinite-dimensional integration technique that physicists use a lot, but that from the mathematical point of view is far from being understood (in a formal way). But it turns out to be a very important tool allowing one to quantize arbitrary systems, ranging from classical mechanical systems to field theories, and allowing one to compute arbitrary integral kernels by integrating against a certain action in an infinite-dimensional space of possible trajectories. To a mathematician that is a bit crazy, but there is more to the story, since that is just the first step, in string theory they allow themselves even to integrate with respect to all possible complex structures, metrics, topologies... So one can not pretend to understand the theory from the point of view of a formal mathematician. One has to choose, either we take for a moment the perspective of physicists, their language and their tools, and try to understand it from inside, to later come back and build mathematical theories. Although many of their tools still don't seem to have a formal mathematical theory, they predict to an incredible precision the results of experiments (for instance in QED), and thus there ought to be some degree of truth behind those ideas, at least at the intuitive level.

4 Conclusions

During this work we have introduced a wide variety of tools to study geometric situations, and we have seen some of their uses in dealing with certain problems and questions. We started from the point of view of XIX'th century mathematics and physics, we have seen how even the notion of what geometry means is not something that is set in stone forever.

The notion of geometry is something that has changed during the years and we believe that this trend is a fundamental part of the history of both mathematics and physics, since the practical development of mathematical theories forces us to rethink again and again our own perspective of the world around us. Riemann, as we saw, introduced a revolutionary paradigm that helped physicists like Einstein develop a geometric description of gravity. Of course General Relativity would have been impossible without Riemann's work, as Newtonian mechanics and gravity would have been impossible without the Euclidean notion of geometry and the development of calculus.

The idea that the universe is something that should be described geometrically dates back to the first geometers of humanity, namely mathematicians in ancient Greece. There is a famous quote by Plutarch about this idea, that he attributes to Plato:

ἀεὶ ὁ θεὸς γεωμετρεῖ (God always geometrizes).

We hope to have convinced the reader that, although this idea might be full of the mysticism of their time, there is a big component of truth in that phrase. And one can interpret it today from a materialist perspective, geometry is a fundamental part of how humanity tries to capture the inner workings of nature. But apart from that, there seems to be something that tells us that geometry is usually the best place to look while seeking for an accurate description. Some millenia after the greeks, physicists during last century were also pushed to geometry for understanding fundamental particles, the so-called Gauge Theories. And even though that seemed not to work at first, since they were not able to describe massive bosons by using Yang-Mills theory, and that doomed the theory for some years, after the discovery of the so-called *Higgs mechanism*, the theory gained importance again. And almost 50 years later experimental evidence was found of the Higgs at the LHC.

From the mathematical perspective, last century's development of mathematical ideas has been comparable or even bigger than the physics counterpart. And even though during the XIX'th century mathematics and physics started to live their own "separate lives", there was not much of a distinction before. This does not mean that one should try to merge both fields by force, since one would be very doomed since both communities are very far apart usually. We hope to have convinced the reader that this big separation is starting to become smaller, and there is a mutual flow of ideas from one field to the other, as we have partly seen during the work.

There are very good motivations for learning what the physicists are doing, and viceversa. Groundbreaking ideas have come in the last years from this, for instance [65], [68],[69],[70],[71], [72]...(a complete list would even be hard to make). One can neither fully understand mathematics from the physical point of view neither physics from the purely mathematical point of view. There are many aspects of modern theoretical physics that are nowadays far from being understood. Math and Physics do not form thus a homogeneous and peaceful set of ideas, and many physical ideas cause an estrangement for us as mathematicians, and the same happens in the reverse direction. Physicists, for instance, regularize many computations by using analytic continuation, and that turns out to produce accurate results in experiments, it may seem crazy, but that's what they do. One could say that they are merely motivated by the experimental result, but neglecting the beauty behind the idea is hiding the *phenomenological* problem under the carpet. The "quest for beauty" in the form of mathematics has been a perspective that many of the great physicists of last centuries have shared, Newton, Dirac, Witten...

Many of the ideas that were developed for a purely mathematical reason later turned out to be of fundamental importance for understanding the physical world, manifolds, sheaves, complex manifolds, group theory...

We are in a time in history where this is no longer a one-directional influence, physical ideas permeate mathematics, and many minds are needed to pursue the research on this topics, and we are seeing an increasing interest in exploring both. A century ago, describing Gravity with Riemannian geometry was one of the biggest revolutions that science had ever seen, it changed forever the way we see the universe around us. The point is that history doesn't stop there, and we are faced today with problems that may be orders of magnitude bigger, getting a good grasp at the inner workings of nature maybe has never been so complicated, a task under construction still today. We hope that we have helped motivate at least the interest of the reader in further exploring the mutual relationship between geometry and modern theoretical physics, two topics that can not be further separated from each other. Maybe this has always been like this, and humanity just realizes from time to time the truth behind Plato's thesis, but that is a philosophical debate that only the future, scientific research and a good historical perspective and analysis will show us, it cannot be settled theoretically but only through the development of both mathematics and physics.

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